Bachelor of Commerce (DDE)

Semester – I

Paper Code – BM1004-1

BUSINESS MATHEMATICS-I

DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

(A State University established under Haryana Act No. XXV of 1975)
NAAC 'A+' Grade Accredited University

Material Production

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Maharshi Dayanand University
ROHTAK – 124 001

ISBN :

Price : Rs. 325/-

Publisher: Maharshi Dayanand University Press

Publication Year : 2021

BUSINESS MATHEMATICS-I
PAPER CODE: BM1004-I

Theory Paper Max Marks: 80
Time: 3Hrs **Internal marks: 20**

Unit I

Calculus: (Problems and theorems involving trigonometrically ratios are not to be done). **Differentiation:** Partial derivatives up to second order; Homogeneity of functions and Euler's theorem; total differentials, Differentiation of implicit function with the help of total differentials. Maxima and Minima; Cases of one variable involving second or higher order derivatives; Cases of two variables involving not more than one constraint.

Unit-II

Integration: Integration as anti-derivative process; Standard forms; Methods of integration-by substitution, by parts, and by use of partial fractions; Definite integration; Finding areas in simple cases; Consumers and producers surplus; Nature of Commodities learning Curve; Leontiff Input-Output Model.

Unit-III

Matrices: Definition of matrix; Types of matrices; Algebra of matrices;

Unit-IV

Determinants: Properties of determinants; calculation of values of determinants up to third order; Adjoint of a matrix, through Adjoint and elementary row or column operations; Solution of system of linear equations having unique solution and involving not more than three variables.

Suggested Readings:

1. *Allen B.G.D: Basic Mathematics; Mcmillan, New Delhi.*
2. *Volra. N. D. Quantitative Techniques in Management, Tata McGraw Hill, New Delhi. Kapoor V.K. Business*
3. *Mathematics: Sultan chand and sons, Delhi.*

Contents

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Calculus and Differentiation

Structure

1.1. Introduction.

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1.9. Check Your Progress.

1.10. Summary.

1.1. Introduction. This chapter contains many important results related derivatives, partial derivatives and their use to obtain extreme values of a function.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Derivatives.

(ii) Partial Derivatives.

(iii) Euler's Theorem.

(iv) Maxima and Minima

1.1.2. Keywords. Continuity, Differentiation, Partial Differentiation, Homogeneous Functions.

1.2. Differentiation.

Differentiation is the technique of determining the derivatives of continuous functions and derivative is the limit of average rate of change in the dependent function following a change in the value of the variable. Very small change in the value of independent variable is accompanied by a very small change in the value of dependent variable.

Mathematically, we say that y is a function of x or y = f(x). The set of all permissible values of x is called **Domain** of the function and the set of corresponding values of y is called the **Range** of the function.

1.2.1. Derivative of a function.

To obtain the derivative of a given function:

Let

$$y = f(x) \qquad (1)$$

be the given function of *x*.

Given a small increment δx in *x,* assume δy be the corresponding increment in *y* so that

$$y + \delta y = f(x + \delta x) \qquad (2)$$

Subtract (1) from (2), we get

$$\delta y = f(x + \delta x) - f(x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Proceeding to limits $\delta x \to 0$ which gives

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

On evaluating the limit of equation (5), we get the value of $\frac{dy}{dx}$

This method of obtaining derivative is known as **differentiation from first principle** or by **ab-initio method** or **from delta method** or **from definition**.

1.2.2. Example. The derivative of x^n is nx^{n-1} where n is fixed number, integer or rational.

Solution. Let

$$y = x^n \qquad (1)$$

Let δx be a small increment in *x* and δy be the corresponding increment in *y,* then

$$y + \delta y = (x + \delta x)^n \qquad (2)$$

Subtracting (1) from (2), we get

$$\delta y = (x + \delta x)^n - x^n$$

dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

Proceeding to limits as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = n.x^{n-1}$$

Using the fact that $\lim_{x \to a} \frac{x^n - a^n}{x-a} = n.a^{n-1}$

Hence $\frac{d}{dx}(x^n) = n.x^{n-1}$

1.2.3. Example. Find the derivatives of the functions

(i) x^{10} (ii) x^{-9} (iii) $x^{2/3}$

Solution.

(i) Let $y = x^{10}$, then $\frac{dy}{dx} = 10.x^9$.

(ii) Let $y = x^{-9}$, then $\frac{dy}{dx} = -9.x^{-10}$.

(iii) Let $y = x^{2/3}$, then $\frac{dy}{dx} = \frac{2}{3}.x^{-\frac{1}{3}}$.

1.2.4. Example. The derivative of $(ax+b)^n$ is $na(ax+b)^{n-1}$

Solution. Let $y = (ax+b)^n$ (1)

Let δx be a small increment in *x* and δy be the corresponding increment in *y,* then

$$y + \delta y = [a(x+\delta x)+b]^n = [(ax+b)+a\delta x]^n \qquad (2)$$

Subtracting (1) from (2), we get

$$\delta y = [(ax+b)+a\delta x]^n - (ax+b)^n$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{\delta x} = a\frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{a\delta x}$$

Proceeding to limits as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} a\frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{a\delta x} = na(ax+b)^{n-1}$$

Hence, $\frac{dy}{dx} = na(ax+b)^{n-1}$.

1.2.5. Example. Find the derivative of $x^2 + 3x$ w.r.t. 'x' by using the first principle.

Solution. Let $y = x^2 + 3x$ (1)

Let δx be small increment in x and δy be the corresponding increment in y, then

$$y + \delta y = (x + \delta x)^2 + 3(x + \delta x) \quad (2)$$

Subtracting (1) from (2), we get

$$\delta y = (\delta x)^2 + 2x\delta x + 3(\delta x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \delta x + 2x + 3$$

Proceeding to limits as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x + 3$$

Hence $$\frac{dy}{dx} = 2x + 3$$

1.2.6. Example. Find the derivative of $x + \frac{1}{x}$ w.r.t. 'x'.

Solution. Let $y = x + \frac{1}{x} = x + x^{-1}$ then, $\frac{dy}{dx} = 1.x^{1-1} + (-1)x^{-1-1} = 1 - \frac{1}{x^2}$.

1.2.7. Example. Differentiate $\sqrt{x} + \frac{1}{x^{\frac{3}{2}}}$ w.r.t. 'x.

Solution. Let $y = \sqrt{x} + \frac{1}{x^{\frac{3}{2}}} = x^{\frac{1}{2}} + x^{-\frac{3}{2}}$, then $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} + \left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{5}{2}}$.

1.2.8. Results.

1. $\frac{d}{dx}(c) = 0$ where c is constant function.
2. $\frac{d}{dx}[c.\, f(x)] = c\, .\frac{d}{dx}[f(x)]$ where 'c' is constant.
3. If u and v are differentiable functions of 'x' then $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$ and $\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$.
4. **Product Rule for differentiation.** If u,v and w are functions of x then

   (i) $\frac{d}{dx}(u.v) = u\frac{d}{dx}v + v\frac{d}{dx}u$

(ii) $\frac{d}{dx}(u.v.w) = vw\frac{d}{dx}(u) + wu\frac{d}{dx}(v) + uv\frac{d}{dx}(w)$

5. **Quotient Rule for Differentiation.** If u and v are functions of x and $v \neq 0$ then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}v}{v^2}.$$

6. **Chain Rule.** If $y = f(u)$ and $u = \phi(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

7. If $y = f(u)$, $u = g(v)$ and $v = \phi(x)$ are three differentiable function, then by chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left[\frac{du}{dv}\frac{dv}{dx}\right] = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

that is, $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}.$$

1.2.9. Example. Differentiate $(x+a)^m (x+b)^n$ w.r.t. x

Solution. Let $y = (x+a)^m (x+b)^n$, then using the product rule of differentiation, we have

$$\begin{aligned}
\frac{dy}{dx} &= (x+a)^m \frac{d}{dx}(x+b)^n + (x+b)^n \frac{d}{dx}(x+a)^m \\
&= (x+a)^m.n(x+b)^{n-1}.1 + (x+b)^n.m(x+b)^{m-1}.1 \\
&= (x+a)^{m-1}.(x+b)^{n-1}\left[n(x+a)+m(x+b)\right] \\
&= (x+a)^{m-1}(x+b)^{n-1}\left[(m+n)x+an+bm\right].
\end{aligned}$$

1.2.10. Example. Differentiate $\frac{ax+b}{cx+d}$ w.r.t. x

Solution. Let $y = \frac{ax+b}{cx+d}$, then by quotient rule of differentiation, we have

$$\frac{dy}{dx} = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{(cx+d)(a+0)-(ax+b)(c+0)}{(cx+d)^2} = \frac{acx+ad-acx-bc}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}.$$

1.2.11. Example. If $y = u^2+u+6$, $u = v^2+75$, $v = 6x+17$, then find $\frac{dy}{dx}$.

Solution. We have $\frac{dy}{du} = 2u+1$, $\frac{du}{dv} = 2v$, $\frac{dv}{dx} = 6$

By chain rule $$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx}=(2u+1)\,(2v)(6)=\left[2\left(v^2+75\right)+1\right]12\,v$$

$$=12\,v\left(2v^2+151\right)=12\left(6x+17\right)\left[2\left(6x+17\right)^2+11\right].$$

1.2.12. Exercise. Find the derivatives of the following functions w.r.t. x by using first principle.

1. x^4 **2.** $9x+10$ **3.** $x^2+10x+80$

4. $x^{\frac{4}{5}}$ **5.** $x^{-\frac{1}{4}}$ **6.** $\frac{2}{\sqrt{x}}$

7. $\frac{x+1}{x^2}$ **8.** $x^{\frac{1}{4}}+\frac{1}{x}$ **9.** $\sqrt{2x+7}$

10. $(7x+6)^{-4}$ **11.** $\frac{ax+b}{cx+d}$ **12.** $2x+\frac{1}{2x^{\frac{3}{2}}}$.

13. If $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$, then show that $\frac{dy}{dx}-y+\frac{x^n}{n!}=0$.

Answers.

1. $4x^3$ **2.** 9 **3.** $2x+10$

4. $\frac{4}{5x^{\frac{1}{5}}}$ **5.** $-\frac{1}{4}x^{-\frac{5}{4}}$ **6.** $-\frac{1}{x^{\frac{3}{2}}}$

7. $-\frac{1}{x^2}-\frac{1}{x^3}$ **8.** $\frac{1}{4x^{\frac{3}{4}}}-\frac{1}{x^2}$ **9.** $\frac{1}{\sqrt{2x+7}}$

10. $-\frac{28}{\left(7x+6\right)^5}$ **11.** $\frac{ad-bc}{(cx+d)^2}$ **12.** $2-\frac{1}{2x^{\frac{5}{2}}}$.

1.2.13. Exercise. Differentiate w.r.t. x the following:

1. $(x^2+1)(x^2+x+4)$ **2.** $x(x-3)(x^2+x)$ **3.** $\frac{x+3}{x^2+1}$

4. $\frac{3x+2}{(x+5)(2x+1)+3}$ **5.** $y=v^3+2v^2+5$, $v=3u+1$ and $u=9x+1$

Answers.

1. $4x^3+3x^2+10x+1$ **2.** $2x(2x^2-3x-3)$ 3. $\frac{1-6x-x^2}{(x^2+1)^2}$

4. $\frac{-6x^2-8x+2}{(2x^2+11x+8)^2}$ 5. $27\left(2187\,x^2+756\,x+64\right)$

1.3. Differentiation of Logarithmic and Exponential functions.

1.3.1. Exponential function. If 'a' be any positive real number then $y = a^x$ is called an exponential function where $x \in R$. When $a = e$, then $y = e^x$ is called exponential function.

1.3.2. Derivative of Exponential function. $\dfrac{d}{dv}(a)^u = a^u \log_e a \dfrac{du}{dv}$.

Also when $a = e$, then $\dfrac{d}{dv}a^x = a^x \log_e a$.

In particular, when $u = x$ then $\dfrac{d}{dx}(e^x) = e^x$.

1.3.3. Derivative of Logarithmic Function.

If u is any differentiable function of x, then

$$\frac{d}{dx}\log_a u = \frac{1}{u}\log_a e\,\frac{d}{dx}(u)$$

In particular, when $u = x$ then $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

1.3.4. Some Properties of Logrithm.

(i) $\log_a(m\,.\,n) = \log_a m + \log_a n$ **(ii)** $\log_a \dfrac{m}{n} = \log_a m - \log_a n$

(iii) $\log_a m^n = n\log_a m$ **(iv)** $\log_a m = \log_b m\,.\,\log_a b$

(v) $\log_b a = \dfrac{\log a}{\log b}$

1.3.5. Example. Differentiate the following functions w.r.t. 'x'

(i) e^{5x+3} **(ii)** e^{e^x} **(iii)** 8^{5x+7}

Solution. (i) Let $y = e^{5x+3}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{5x+3} = e^{5x+3}\dfrac{d(5x+3)}{dx} = 5e^{5x+3}$.

(ii) Let $y = e^{e^x}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{e^x} = e^{e^x}\dfrac{de^x}{dx} = e^x e^{e^x}$.

(iii) Let $y = 8^{5x+7}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(8^{5x+7}) = 8^{5x+7}\dfrac{d(5x+7)}{dx}\log 8 = (5\log 8)8^{5x+7}$.

1.3.6. Exercise. Differentiate the following functions w.r.t. 'x':

1. $\log\left(x + \sqrt{a^2 + x^2}\right)$ **2.** $\log\left[\log\left(\log x^4\right)\right]$ **3.** $e^x \log\left(1 + x^2\right)$

4. **(i)** If $y=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$, then prove that $\dfrac{dy}{dx}=1-y^{2}$

(ii) If $y=\log\left(\sqrt{x-1}-\sqrt{x+1}\right)$, then prove that $\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{x^{2}-1}}$

(iii) If $y=(x-1)\log(x-1)-(x+1)\log(x+1)$, then prove that $\dfrac{dy}{dx}=\log\left(\dfrac{x-1}{x+1}\right)$.

Answer.

1. $\dfrac{1}{\sqrt{a^{2}+x^{2}}}$ 2. $\dfrac{4}{x\left(\log x^{4}\right)\left[\log\left(\log x^{4}\right)\right]}$ 3. $e^{x}\left[\dfrac{2x}{1+x^{2}}+\log\left(1+x^{2}\right)\right]$

1.4. Partial derivatives.

Let f be a function of two or more variables. The derivative of f w.r.t. one independent variable, while considering all other independent variables constant, is called the partial derivative of f w.r.t. that variable.

If $f(x,y)$ is a function of two independent variables x and y, then the partial derivative of $f(x,y)$ w.r.t. x is the derivative of $f(x,y)$ when y is regarded as constant. It is denoted by $\dfrac{\partial f}{\partial x}$ or f_x or $D_x f$. Thus,

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}.$$

Similarly, the partial derivative of $f(x,y)$ w.r.t. y is defined as $\dfrac{\partial f}{\partial y}=\lim\limits_{k\to 0}\dfrac{f(x,y+k)-f(x,y)}{k}$.

This definition can be extended to a function of having more than two independent variables.

1.4.1. Second order partial derivatives. If $f(x,y)$ has partial derivatives at each point, then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are themselves functions of x and y, which may also have partial derivatives, known as second order partial derivatives. These second derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^{2} f}{\partial x^{2}}=f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^{2} f}{\partial y\,\partial x}=f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^{2} f}{\partial x\,\partial y}=f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^{2} f}{\partial y^{2}}=f_{yy}$$

1.4.2. Remark. Generally $\dfrac{\partial^2 f}{\partial y\,\partial x} \neq \dfrac{\partial^2 f}{\partial x\,\partial y}$. But here we will deal with only those functions for which $\dfrac{\partial^2 f}{\partial y\,\partial x} = \dfrac{\partial^2 f}{\partial x\,\partial y}$. Also, all the results stated for ordinary differentiation, like chain rule, product rule, quotient rule etc., are valid for partial differentiation.

1.4.3. Example. Find the all the first and second order partial derivatives of $x^5 + y^5 - 2ax^2 y + xy$.

Solution. Let $u = x^5 + y^5 - 2ax^2 y + xy$, then

$$\frac{\partial u}{\partial x} = 5x^4 - 4axy + y, \qquad \frac{\partial u}{\partial y} = 5y^4 - 2ax^2 + x$$

$$\frac{\partial^2 u}{\partial x^2} = 20x^3 - 4ay, \qquad \frac{\partial^2 u}{\partial y^2} = 20y^3$$

$$\frac{\partial^2 u}{\partial x \partial y} = -4ax + 1, \qquad \frac{\partial^2 u}{\partial y \partial x} = -4ax + 1$$

1.4.4. Exercise.

1. If $u = \log\left(x^2 + y^2\right)$, then prove that

   $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$

2. For the function $z = x^y + y^x$ verify that $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$.

3. If $u = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$.

4. If $u = 3\left(lx + my + nz\right)^2 - \left(x^2 + y^2 + z^2\right)$ and $l^2 + m^2 + n^2 = 1$. Show that

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

5. Find the first and second order partial derivative of $y\log x$.

6. If $z^3 - xz - y = 0$, prove that $\dfrac{\partial^2 z}{\partial x\,\partial y} = -\dfrac{3z^2 + x}{\left(3z^2 - x\right)^3}$.

7. Find the value of $\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$ when $a^2x^2 + b^2y^2 - c^2z^2 = 0$.

8. If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \left(1 + 3xyz + x^2 y^2 z^2\right) e^{xyz}$. If $u = f\left(\dfrac{y}{x}\right)$, show that

   $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

1.5. Total Differentials.

1.5.1. Total differential of a function. If $z = f(x, y)$, then the total differential of z is defined by $dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$ and is denoted by dz.

Now we will try to obtain total derivative of a given composite function.

1.5.2. Theorem. Let $z = f(x, y)$ be a function having continuous first order partial derivatives and $x = \phi(t),\ y = \psi(t)$ have continuous derivatives. Then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

Proof. Let t be given a small increment δt and the corresponding changes in z, x and y be δz, δx and δy respectively. Then, we have

$$z + \delta z = f(x + \delta x,\ y + \delta y)$$

$$\Rightarrow \delta z = f(x + \delta x,\ y + \delta y) - f(x, y)$$

$$\Rightarrow \delta z = \left[f(x + \delta x,\ y + \delta y) - f(x,\ y + \delta y)\right] + \left[f(x, y + \delta y) - f(x\ ,\ y)\right]$$

$$\Rightarrow \frac{\delta z}{\delta t} = \left\{\frac{f(x + \delta x,\ y + \delta y) - f(x, y + \delta y)}{\delta t}\right\} + \left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta t}\right\}$$

$$= \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\right\}\frac{\delta x}{\delta t} + \left\{\frac{f(x, y + \delta y - f(x, y)}{\delta y}\right\}\frac{\delta y}{\delta t} \qquad (1)$$

Let $\delta t \to 0$, so that $\delta x \to 0$ and $\delta y \to 0$.

Now, $\lim\limits_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$

and $\lim\limits_{\delta x \to 0} \frac{f(x, y + \delta y) - f(x, y + \delta y)}{\delta y} = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$

Since δx and δy are increments in x, y corresponding to t, therefore, both tend to zero as δt tends to zero. Due to the continuity of f and its partial derivatives (1) becomes

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

1.5.3. Remark.

1. Here $\frac{dz}{dt}$ is called total derivative of z.
2. If $z = f(x_1, x_2, ..., x_n)$ where $x_1, x_2, ..., x_n$ are all functions of some variable t, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt}.$$

3. If $z = f(x, y)$, and $y = f(x)$, then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

4. If $z = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t_1}$$

$$\frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t_2}$$

1.5.4. Example. If $z = xy^2 + x^2 y,\ x = at^2\ ,\ y = 2at$, then find $\frac{dz}{dt}$.

Solution. Since $z = xy^2 + x^2 y \ \Rightarrow\ \frac{\partial z}{\partial x} = y^2 + 2xy\ ,\ \frac{\partial z}{\partial y} = 2xy + x^2$

Also, $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$. Thus,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(y^2 + 2xy\right)2at + \left(2xy + x^2\right)2a$$

$$= \left(4a^2t^2 + 4a^2t^3\right)2at + \left(4a^2t^3 + a^2t^4\right)2a$$

$$= a^3\left(16t^3 + 10t^4\right)$$

1.6. Implicit Functions. If x and y are connected by a functional relation $f(x, y) = c$, then this is called implicit function of x and y.

Consider $f(x, y) = c$. To find $\frac{dy}{dx}$, first express y in terms of x and then differentiate w.r.t. x. However, in case of implicit functions it is impossible to express y in terms of x, we will use the forthcoming method to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Method to Find $\frac{dy}{dx}$.

Let $u = f(x, y)$ be a function of x and y, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad (1)$$

If we are given an implicit function of x and y of the form

$$u = f(x, y) = c$$

Then. $$\frac{du}{dx} = 0.$$

Hence from (1), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

Or $$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

Method to Find $\frac{d^2 y}{dx^2}$.

Since $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad (2)$$

Denote $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ by p, q, r, s and t respectively. Then equation (2) becomes

$$\frac{dy}{dx} = -\frac{p}{q}$$

Using product rule of differentiation

$$\frac{d^2 y}{dx^2} = -\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2} \qquad (3)$$

But $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{dy}{dx} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{dx} = r + s\left(-\frac{p}{q}\right)$$

that is, $$\frac{dp}{dx} = \frac{qr - ps}{q}$$

Also $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\frac{dy}{dx} = \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx} = s + t\left(-\frac{p}{q}\right)$$

that is, $$\frac{dq}{dx} = \frac{sq - pt}{q}$$

Using these in (3), we obtain

$$\frac{d^2 y}{dx^2} = -\frac{q^2 r - 2pqs + p^2 t}{q^3}$$

which can be written as: $$\frac{d^2y}{dx^2} = -\frac{f_{xx}\, f_y^2 - 2f_{xy}\, f_x\, f_y + f_{yy}\, f_x^2}{f_y^3}.$$

1.6.1. Example. If $y^3 - 3ax^2 + x^3 = 0$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution. Let $f(x, y) = y^3 - 3ax^2 + x^3 = 0$

Then, $f_x = -6a + 3x^2$, $f_{xx} = -6a + 6x$, $f_y = 3y^2$, $f_{yy} = 6y$, $f_{xy} = 0$.

Now, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{6ax + 3x^2}{3y^2} = \frac{2ax - x^2}{y^2}$, and

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}\, f_y^2 - 2f_{xy}\, f_x\, f_y + f_{yy}\, f_x^2}{f_y^3}$$

$$= -\frac{(-6a + 6x)(9y^4) - 2(-6ax + 3x^2)(3y^2)(0) + 6y(-6ax + 3x^2)^2}{(3y^2)^3}$$

$$= -\frac{-54ay^4 + 54xy^4 + 6y(36a^2x^2 + 9x^4 - 36ax^3)}{27y^6}$$

$$= -\frac{54y[-ay^3 + xy^3 + 4a^2x^2 + x^4 - 4ax^3]}{27y^6}$$

$$= -\frac{2}{y^5}\left[-ay^3 + xy^3 + 4a^2x^2 + x^4 - 4ax^3\right]$$

Replacing $y^3 = 3ax^2 - x^3$ in numerator, we get

$$\frac{d^2y}{dx^2} = -\frac{2}{y^5}\left[-a(3ax^2 - x^3) + x(3ax^2 - x^3) + 4a^2x^2 + x^4 - 4ax^3\right]$$

$$= -\frac{2}{y^5}\left[a^2x^2\right] = -\frac{2a^2x^2}{y^5}$$

1.6.2. Example. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y}\ \frac{\partial \phi}{\partial z}\ \frac{dz}{dx} = \frac{\partial f}{\partial x}\ \frac{\partial \phi}{\partial y}$.

Solution. If $f(x, y) = 0$, then $\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$ and for $\phi(y, z) = 0$, then $\frac{dz}{dy} = -\frac{\left(\frac{\partial \phi}{\partial y}\right)}{\left(\frac{\partial \phi}{\partial z}\right)}$.

Multiplying both, we have

$$\frac{dy}{dx} . \frac{dz}{dy} = \frac{\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)}{\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial \phi}{\partial z}\right)}$$

or $$\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial \phi}{\partial z}\right).\frac{dz}{dx}=\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)$$

1.6.3. Exercise.

1. Find $\frac{dz}{dt}$, when $z = xy^2 + x^2 y$; $x = at^2$, $y = 2at$.
2. If $u = f(y-z, z-x, x-y)$, show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.
3. If $z = u^2 + v^2 + w^2$, where $u = ye^x$, $v = xe^{-y}$, $w = \frac{y}{x}$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
4. Let $z = f(x, y)$ and u, v are two variables given by $u = lx + my$, $v = ly - mx$ show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(l^2+m^2\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$.
5. If $x = u^2 - v^2$, $y = 2uv$ find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.
6. If $x = u + e^{-v}\sin u$, $y = v + e^{-v}\cos u$ prove that $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$.
7. Find $\frac{dy}{dx}$ if $x^y + y^x = a^b$.
8. Find $\frac{d^2 y}{dx^2}$ if $x^5 + y^5 - 5a^3 xy = 0$.
9. Find $\frac{d^2 y}{dx^2}$ if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
10. If $x^3 + y^3 - 3axy = 0$, find $\frac{d^2 y}{dx^2}$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.
11. If $z = xy\, f\left(\frac{y}{x}\right)$ and z is a constant, then show that $\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)}=\frac{x\left[y+x\frac{dy}{dx}\right]}{y\left[y-x\frac{dy}{dx}\right]}$.

Answers.

1. $2a^3 t^3\ (8+5t)$

3. $2y^2 e^{2x}+2xe^{-2y}-\frac{2y^2}{x^3}$; $2y\, e^{2x}-2x^2 e^{-2y}+\frac{2y}{x^2}$

5. $\frac{u}{2\left(u^2+v^2\right)}$; $\frac{-v}{2\left(u^2+v^2\right)}$; $\frac{v}{2\left(u^2+v^2\right)}$; $\frac{u}{2\left(u^2+v^2\right)}$

7. $\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

8. $-\frac{6a^3 xy\left(x^3 y^3+2a^6\right)}{\left(y^4-a^3 x\right)^3}$

9. $\frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}\ y^{\frac{1}{3}}}$

10. $\frac{-32}{3a}$

1.7. Homogeneous Functions. A polynomial in x and y such that the degree of each terms is same is called a homogeneous function of degree n. It can be represented as

$$f(x,y) = a_0x^n + a_1x^{n-1}y + \ldots + a_{n-1}x\,y^{n-1} + a_ny^n$$

A homogeneous function of degree n can be expressed as $x^n\ \phi\left(\frac{y}{x}\right)$. Thus,

$$\begin{aligned} f(x,y) &= a_0x^n + a_1x^{n-1}y + \ldots + a_{n-1}xy^{n-1} + a_ny^n \\ &= x^n\left[a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \ldots + a_n\left(\frac{y}{x}\right)^n\right] \\ &= x^n\phi\left(\frac{y}{x}\right) \end{aligned}$$

For example, if $f(x,y) = \dfrac{\sqrt{y}+\sqrt{x}}{y+x} = \dfrac{\sqrt{x}\left[1+\sqrt{\frac{y}{x}}\right]}{x\left[1+\frac{y}{x}\right]} = x^{-\frac{1}{2}}\ \phi\left(\frac{y}{x}\right)$.

Therefore $f(x, y)$ is a homogeneous function of degree $-\dfrac{1}{2}$.

1.7.1. Theorem. If u is a homogeneous function in x and y of degree n, then show that $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are homogeneous functions of degree $(n-1)$ each.

Proof. Given that u is a homogeneous function of x and y of degree n, so by definition, we can write $u = x^n\ f\left(\frac{y}{x}\right)$, thus

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(x^n\right)f\left(\frac{y}{x}\right) + x^n\frac{\partial}{\partial x}\left[f\left(\frac{y}{x}\right)\right] = nx^{n-1}f\left(\frac{y}{x}\right) + x^nf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) \\ &= n\,x^{n-1}\ f\left(\frac{y}{x}\right) + x^{n-1}\ f'\left(\frac{y}{x}\right)\left(-\frac{y}{x}\right) = x^{n-1}\left[\text{some function of } \frac{y}{x}\right] \\ &= x^{n-1}\phi\left(\frac{y}{x}\right) \end{aligned}$$

This shows that $\dfrac{\partial u}{\partial x}$ is a homogeneous function of degree $n-1$

Also,

$$\begin{aligned} \frac{\partial u}{\partial y} &= x^n\ \frac{\partial}{\partial y}\left[f\left(\frac{y}{x}\right)\right] = x^n\ f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1}\ f'\left(\frac{y}{x}\right) \\ &= x^{n-1}\left[\text{a function of } \frac{y}{x}\right] \end{aligned}$$

Hence $\frac{\partial u}{\partial y}$ is also a homogeneous function of degree $n - 1$.

1.7.2. Euler's Theorem. If $u = f(x, y)$ be a homogeneous function of x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

for all x, y belonging to the domain of the function.

Proof. As u is a homogeneous function of degree n, Therefore,

$$u = x^n \phi\left(\frac{y}{x}\right) \qquad (1)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^n\right)\phi\left(\frac{y}{x}\right) + x^n \frac{\partial}{\partial x}\left[\phi\left(\frac{y}{x}\right)\right]$$

$$= nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = n\,x^{n-1}\phi\left(\frac{y}{x}\right) - x^{n-2}\,y\,\phi'\left(\frac{y}{x}\right) \qquad (2)$$

Again $$\frac{\partial u}{\partial y} = x^n \frac{\partial}{\partial y}\left[\phi\left(\frac{y}{x}\right)\right] = x^n \phi'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1}\phi'\left(\frac{y}{x}\right) \qquad (3)$$

Multiplying (2) by x, (3) by y and adding, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) - x^{n-1}\,y\,\phi'\left(\frac{y}{x}\right) + x^{n-1}y\,\phi'\left(\frac{y}{x}\right)$$

$$= n\,x^n \phi\left(\frac{y}{x}\right) = nu \qquad \text{[By (1)]}$$

1.7.3. Remark. In general, if u is a homogeneous function of m independent variables, $u = u\left(x_1, x_2, ..., x_m\right)$, then $x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + ... + x_m\frac{\partial u}{\partial x_m} = nu$.

1.7.4. Theorem. If u is a homogeneous function of x and y of degree n, then show that

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x},$$

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

and $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof. Since u is homogeneous function of x, y of degree n, so by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad (1)$$

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(nu)$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}. \qquad (2)$$

Similarly, differentiating (1) partially w.r.t. y, we have

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y} \qquad (3)$$

Multiplying (2) by x and (3) by y and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = (n-1)\,nu$$

Using Euler's theorem, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

1.7.5. Example. If $u = x^2 + y^2$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

Solution. Since $u = x^2 + y^2$, so $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 2y$. Thus,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

1.7.6. Example. If $z = x^m f\left(\frac{y}{x}\right) + x^n\ g\left(\frac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mn^2 = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Solution. Let $u = x^m\ f\left(\frac{y}{x}\right)$ and $v = x^n\ g\left(\frac{x}{y}\right)$. Then $z = u + v$.

Now $u = x^m\left(\frac{y}{x}\right)$, so using Euler's theorem, we have

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = m(m-1)u \qquad (1)$$

Also $v = x^n g\left(\frac{x}{y}\right)$, so

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v \qquad (2)$$

Adding (1) and (2), we have

$$x^2 \frac{\partial^2}{\partial x^2}(u+v) + 2xy \frac{\partial^2}{\partial x \partial y}(u+v) + y^2 \frac{\partial^2}{\partial y^2}(u+v) = m(m-1)u + n(n-1)v$$

$$\begin{aligned} x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} &= m(m-1)u + n(n-1)v \\ &= (m^2u + n^2v) - (mu + nv) \\ &= m(m+n)u + n(m+n)v - mn(u+v) - (mu+nv) \\ &= (m+n)(mu+nv) - mn(u+v) - (mu+nv) \\ &= (m+n-1)(mu+nv) - mn(z) \end{aligned}$$

Also due to Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu \quad \text{and} \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

and so

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = mu + nv$$

as $z = u + v$. Thus,

$$x^2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (m+n-1)\left(x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - mnz$$

or $$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right).$$

1.7.7. Exercise. Verify Euler's theorem for the following function:

1. $u = xy\, f\left(\frac{y}{x}\right)$
2. $u = 2x^5 - 3x^4y + 5x^2y^3 + 3xy^4 - 7y^5$
3. $u = e^{\frac{y}{x}}$
4. $u = x^n \log\left(\frac{y}{x}\right)$.

1.8. Local Maxima and Local Minima.

1.8.1. Local Maxima. A function $f(x)$ is said to have a local maxima at $x = a$ if there exists $\delta > 0$ such that

$$f(x) < f(a), \text{ for all } x \in (a-\delta,\ a+\delta)$$

Here, $f(a)$ is known as local maximum value of the function $f(x)$ at $x = a$.

1.8.2. Local Minima. A function $f(x)$ is said to have a local minima at $x = a$ if there exists $\delta > 0$ such that

$$f(x) > f(a)\text{, for all } x \in (a-\delta,\ a+\delta)$$

Here, $f(a)$ is known as local minimum value of the function $f(x)$ at $x = a$.

1.8.3. Maximum value of a function. Let $f(x)$ is real valued function on the interval I, then $f(x)$ is said to have the Maximum value in I, if there exists some a in I such that

$$f(x) \le f(a), \text{ for all } x \in I.$$

A $f(x)$ having Maximum value at $x = a$, if there exists a neighbourhood of $x = a$ such that it is an increasing function on left hand side of $x = a$ and decreasing function on the right hand side of $x = a$.

1.8.4. Minimum value of a function. Let $f(x)$ is real valued function on the interval I. then $f(x)$ is said to have the minimum value in I, if there exists some a in I such that

$$f(x) \ge f(a)\text{, for all } x \in I$$

A $f(x)$ having Minimum value at $x = a$, if there exists a neighbourhood of $x = a$ such that it is a decreasing function on left hand side of $x = a$ and increasing function on the right hand side of $x = a$.

1.8.5. Stationary Point. The values of x for which $f'(x) = 0$ are called stationary points or critical points of $f(x)$.

1.8.6. Theorem. A necessary condition for $f(a)$ to be an extreme value of function $f(x)$ is that $f'(a) = 0$, if exists.

1.8.7. First derivative test to find points of local maxima and local minima.

First assume $y = f(x)$ be a differentiable function. Then, differentiate y with respect to x and solve $\frac{dy}{dx} = 0$ for x. If $c_1, c_2, ..., c_n$ be the roots of this equation, then these are the possible points (known as stationary points) where the function can attain a local maxima or local minima. When $x = c_i$ and $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_i, then the function attains a local maxima at $x = c_i$ and **local maximum value** at $x = c_i$ is $f(c_i)$. Further, if $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_i, then the function attains a local minima at $x = c_i$ and

local minimum value at $x = c_i$ is $f(c_i)$. If $\frac{dy}{dx}$ does not change sign as x increases through c_i**,** then $x = c_i$ is neither a point of Local maximum nor a point of local minimum. In this case $x = c_i$ is a **point of inflexion.**

1.8.8. Higher order derivative test to find points of local maxima and local minima.

First assume $y = f(x)$ be a differentiable function. Then differentiate w.r.t. x to find $f'(x)$. Solve $f'(x) = 0$ for x and let $c_1, c_2, ..., c_n$ be the roots of this equation, then these are the possible points (known as stationary points) where the function can attain a local maximum or a local minimum. At $x = c_i$, if $f''(c_i) < 0$, then $x = c_i$ is a point of local maximum and local maximum value is $f(c_i)$. Similarly, if $f''(c_i) > 0$, then $x = c_i$ is a point of local minimum and local minimum value is $f(c_i)$.

If $f''(c_i) = 0$ and $f'''(c_i) \neq 0$, then $x = c_i$ is neither a point of local maxima nor a point of local minima and is called the point of inflexion. However, if $f'''(c_i) = 0$, then for $f^{iv}(c_i) < 0$, then $x = c_i$ is a point of local maximum and local maximum value is $f(c_i)$ and for $f^{iv}(c_i) > 0$, then $x = c_i$ is a point of local minimum and local minimum value is $f(c_i)$.

If $f^{iv}(c_i) = 0$, then proceed to higher derivative as in the case $f''(c_i) = 0$.

1.8.9. Absolute Maxima and Absolute Minima. If a function $f(x)$ is continuous and differentiable on a closed interval $[a, b]$, then it attains the absolute maximum and absolute minimum at the stationary points or at a or b.

At the stationary points $c_1, c_2, ..., c_n$ and a, b obtain $f(c_1), f(c_2), ..., f(c_n), f(a), f(b)$. Out of these values the maximum and minimum values are respectively known as the absolute maximum and absolute minimum values of the function.

1.8.10. Example. Find all the points of local maxima and minima of $f(x) = (x-1)(x+2)^2$ using first derivative test. Also, find local maximum and local Minimum values.

Solution. Let $y = (x-1)(x+2)^2$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2(x-1)(x+2) + (x+2)^2 = 3x(x+2)$$

Considering $\frac{dy}{dx} = 0$, implies $3x(x+2) = 0$ and so $x = 0, -2$.

Therefore, $x = 0$ and $x = -2$ are the critical points.

For $x = 0$, if x is slight less than 0, then

$$\frac{dy}{dx} = 3x(x+2) < 0$$

as $x < 0$ and $x + 2 > 0$. If x is slightly greater than 0, then

$$\frac{dy}{dx} = 3x(x+2) > 0$$

as x > 0 and x + 2 > 0.

Therefore $\frac{dy}{dx}$ changes sign from negative to positive as x passes through 0. Hence $x = 0$ is point of local minima and local minimum value is $f(0) = (0-1)(0+2)^2 = -4$.

For $x = -2$, if x is slightly less than – 2, then

$$\frac{dy}{dx} = 3(x)(x+2) > 0$$

and when x is slightly greater than – 2

$$\frac{dy}{dx} = 3x(x+2) < 0$$

Therefore, $\frac{dy}{dx}$ changes sign from +ve to –ve as x passes through – 2, which implies $x = -2$ is the point of local maxima and local maximum value is $f(-2) = (-2-1)(-2+2)^2 = 0$

1.8.11. Example. Determine the local maximum and local minimum values, if any, of $x^3 - 6x^2 + 9x + 15$**.**

Solution. Consider $f(x) = x^3 - 6x^2 + 9x + 15$

Differentiating $f(x)$ w.r.t. x, we obtain $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$.

For stationary value, $f'(x) = 0 \quad \Rightarrow \quad 3x^2 - 12x + 9 = 0 \quad \Rightarrow \quad 3(x^2 - 4x + 3) = 0$

$$\Rightarrow \quad (x-1)(x-3) = 0 \quad \Rightarrow \quad x = 1, 3$$

Therefore, $x = 1$ and $x = 3$ are the critical points.

For $x = 1$, $f''(1) = 6(1) - 12 = -6 < 0$. Hence $x = 1$ is the point of local maximum and local maximum value is $f(1) = 19$.

For $x = 3$, $f''(3) = 6(3) - 12 = 6 > 0$. Hence $x = 3$ is the point of local minimum and local minimum value is $f(3) = 15$.

1.8.12. Example. Find the absolute maximum and absolute minimum values of $\left(\frac{1}{2} - x\right)^2 + x^2$ on the interval $\left[-2, \frac{5}{2}\right]$.

Solution. Let $f(x) = \left(\frac{1}{2} - x\right)^2 + x^3$, then $f'(x) = 2x - 1 + 3x^2$.

Now $f'(x) = 0 \quad \Rightarrow \quad 3x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{3}, 1$

Since $\frac{1}{3}, -1 \in \left[-2, \frac{5}{2}\right]$, therefore, $x = \frac{1}{3}$ and $x = -1$ are the only stationary points. So we need to find the value of $f(x)$ at $x = -2, -1, \frac{1}{3}, \frac{5}{2}$.

Here, $f(-2) = -\frac{7}{4}$, $f(-1) = \frac{5}{4}$, $f\left(\frac{1}{3}\right) = \frac{7}{108}$, $f\left(\frac{5}{2}\right) = \frac{157}{8}$.

Hence, absolute maximum value of $f(x)$ is $\frac{157}{8}$ at $x = \frac{5}{2}$ and absolute minimum value is $-\frac{7}{4}$ at $x = -2$.

1.8.13. Remarks.

1. Area and parameter of a rectangle of sides x and y are xy and $2(x + y)$.
2. Area and parameter of a square of side x are x^2 and $4x$.
3. Area and circumference of a circle of radius r are πr^2 and $2\pi r$.
4. Volume and Surface area of a cube of edge length x are x^3 and $6x^2$.
5. Volume and Surface area of a cuboid of edges of length x, y and z are xyz and $2(xy + yz + zx)$.
6. Volume and Surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$.
7. Volume, Surface area and Curved Surface area of a right circular cylinder of base radius r and height h are $\pi r^2 h$, $2\pi rh + 2\pi r^2$ are $2\pi rh$ respectively.
8. Volume, Surface area and Curved Surface area of a right circular cone of height h, slant height l and radius of base r are $\frac{1}{3}\pi r^2 h$, $\pi rl + \pi r^2$ and πrl respectively.

1.8.14. Example. Divide 30 into two parts such that their product is maximum .

Solution. Let one part is x. Then, second part will be $30 - x$. Let the product of two parts be P. Then,

$$P = x(30 - x)$$

$$\Rightarrow \qquad P = 30x - x^2 \quad \Rightarrow \quad \frac{dP}{dx} = 30 - 2x$$

For stationary points, we take

$$\frac{dP}{dx} = 0 \quad \Rightarrow \quad 30 - 2x = 0 \quad \Rightarrow \quad x = 15$$

Now, $\frac{d^2P}{dx^2} = -2 < 0$ when $x = 15$. Therefore, P is maximum when $x = 15$, that is, P is maximum when first part is 15 then second part is also 15.

1.8.15. Example. Find two positive numbers with sum 35 and product of square of one and fifth power of second is maximum.

Solution. Assume the numbers are x and y. Then, we have,

$$x + y = 35$$

Let $P = x^2y^5$ or $P = x^2y^5 = (35-y)^2y^5$

Then, $\dfrac{dP}{dy} = (35-y)(y^4)[175-7y] = 7(35-y)y^4(25-y)$

For stationary points, we have

$$\frac{dP}{dy} = 0 \quad \Rightarrow \quad 7(35-y)\ y^4\ (25-y) = 0 \quad \Rightarrow \quad y = 0, 25, 35$$

But $y = 0$ and $y = 35$ are not possible. So $y = 25$. Also,

$$\frac{d^2P}{dy^2} = -7y^4(25-y) + 28(35-y)y^3(25-y) - 7(35-y)y^4.$$

At $y = 25$, $\dfrac{d^2P}{dy^2} = -70(25)^4 < 0$.

Thus, P is maximum when $y = 25$ and so $x = 35 - 25 = 10$.

1.8.16. Example. Show that all the rectangles with a given perimeter, the square has the largest area.
Solution. Let x and y be the lengths of two sides of a rectangle of fixed perimeter P and let A be its area. Then, we have $P = 2(x+y)$ and $A = xy$.

Now, $P = 2(x+y) \Rightarrow y = \dfrac{P}{2} - x$. Then, $A = xy = x\left(\dfrac{P}{2} - x\right) = \dfrac{P}{2}x - x^2$, and so

$$\frac{dA}{dx} = \frac{P}{2} - 2x \quad \text{and} \quad \frac{d^2A}{dx^2} = -2$$

For stationary point, take $\dfrac{dA}{dx} = 0 \Rightarrow x = \dfrac{P}{4}$

For $x = \dfrac{P}{4}$, $\dfrac{d^2A}{dx^2} = -2 < 0$. Therefore, A is maximum when $x = \dfrac{P}{4}$. Also, we have $y = \dfrac{P}{4}$.

Hence A is maximum when $x = y = \dfrac{P}{4}$, that is, when rectangle is a square.

1.8.17. Exercise. Determine the local maximum and local minimum values, if any, for the following functions:

(i) $f(x) = x^3 - 6x^2 + 9x - 8$ **(ii)** $f(x) = \dfrac{x^4}{x-1}$, $x \neq 1$

(iii) $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$ **(iv)** $f(x) = (x-1)^3(x+1)^2$

(v) $f(x) = x + \dfrac{1}{x}$ **(vi)** $f(x) = \dfrac{4}{x+2} + x$

(vii) $f(x) = (x-3)^4$

Answers. (i) $x = 1$ is a point of local maxima and local maximum value is -4, and $x = 3$ is a point of local minima and local minimum value is -8.

(ii) $x = 0$ is the point of local maxima and local maximum value is 0, and $x = \frac{4}{3}$ is the point of local minima and local minimum value is $\frac{256}{27}$.

(iii) $x = \frac{1}{2}$ is the point of local maxima and local maximum value is $\frac{39}{16}$, and $x = 1, -1$ is the point of local minima and local minimum value is 2, -6 respectively.

1.8.18. Exercise. Prove that following functions do not have maxima or minima:

(i) $f(x) = e^{ax+b}$ **(ii)** $f(x) = \log(2x+5)$

1.8.19. Exercise.

1. Find the absolute maximum value and the absolute minimum value of the following functions:

(i) $f(x) = x - x^2$ in $[-2, 5]$ **(ii)** $f(x) = (x-2)\sqrt{x-1}$ in $[1, 10]$

(iii) $f(x) = x^3 - 12x + 551$ in $[-3, -1]$ **(iv)** $f(x) = x^3 - 12x^2 + 18$ in $[1, 10]$

1.8.20. Exercise.

1. Among all pairs of positive numbers with product 256, find those having minimum sum.
2. Find two positive numbers with sum 16 and the sum of whose squares in minimum.
3. Show that of all the rectangles of given area, the square has the smallest perimeter.
4. Show that of all the rectangles inscribed in a given circle, the square has the maximum area.
5. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. Find the side of the square to be cut off so that the volume of the box is maximum possible.
6. Show that for a cone of given volume, curved surface area will be minimum when the height is $\sqrt{2}$ times the radius of the base.
7. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'a' is $\frac{2a}{\sqrt{3}}$.
8. A box with a square top and bottom is to be made to contain 500cc. Material for top and bottom costs Rs. 4 per sq. unit and the material for sides costs Rs. 1 per sq. cm. What is the cost of the least expensive box that can be made?
9. Find two numbers whose sum is 24 and whose product is maximum.
10. Find two numbers x and y such that $x + y = 60$ and xy^3 is maximum.
11. Prove that the area of a right angled triangle of given hypotenuse in maximum when the triangle is isosceles.

12. Show that a cylinder of a given volume which is open at the top, has minimum total surface area if its height is equal to the radius of its base.

13. Show that the height of a cylinder, which is open at the top, having a given surface and maximum volume, is equal to the radius of the base.

14. The cost C of manufacturing an article is given by the formula $C = 3x^2 + \frac{48}{x} + 5$ where x is the number of articles manufactured. Find the minimum value of C.

15. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 41 - 24x - 18x^2$.

Answers.

1.	Both parts 16.	**2.**	Both 8.	**5.**	The side of the squares is 5 cm.
8.	Rs. 600	**9.**	12, 12	**10.**	x = 15, y = 45
16.	49				

1.8.21. Finding Maxima and Minima in cases of two variables involving not more than one constraint.

If $f(x,y)$ be a function of two independent variables x and y. Then $f(x,y)$ is said to have maximum or minimum value at the point (a, b) if $f(a,b) > f(a+h,b+k)$ or $f(a,b) < (a+h,b+k)$ for small values of h and k, positive or negative.

1.8.22. Remark. Maximum or minimum value of a function $f(x,y)$ is called its extreme value. For an extreme value at (a,b), the difference $f(a+h,b+k) - f(a,b)$ must have the same sign for all values of h & k

1.823. Necessary conditions for the function $f(x,y)$ to have an extreme value at (a,b).

Due to Taylor's theorem for function of two variables

$$f(a+h,b+k) - f(a,b) = h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$$

$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\,\frac{\partial^2 f}{\partial x \partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right] + \ldots$$

Now h and k are small enough, so second and higher degree terms of h and k may be neglected. Thus the sign of $f(a+h,\, b+k) - f(a,b)$ will be similar to that of $h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$. For having an extreme value, $f(a+h\,,\; b+k) - f(a,b)$ must have the same sign for all small values of h and k, positive or negative. This is possible only when $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$. Thus the necessary conditions for $f(x,y)$ to have extreme value at (a,b) are

$$\frac{\partial f}{\partial x}(a,b)=0=\frac{\partial f}{\partial y}(a,b)$$

1.8.24. Stationary point. The points satisfying the condition $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ are called stationary point of the function $f(x,y)$.

Saddle point. The stationary point of function $f(x,y)$, is the point obtained from $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ at which the function has neither maximum value nor minimum value is called saddle point of $f(x,y)$.

1.8.25 Condition for a function (*x*, *y*) to have maximum or minimum at a point.

The Taylor's theorem for function of two variables is given by

$$f(a+h\ ,b+k)-f(a,b)=h\frac{\partial f}{\partial x}(a,\ b)+k\ \frac{\partial f}{\partial y}(a,b)$$
$$+\frac{1}{2!}\left[\ h^2\frac{\partial^2 f}{\partial x^2}(a,b)\ +2hk\frac{\partial^2 f}{\partial x\partial y}(a,\ b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\ \right]+...$$

For $f(x,y)$ to have extreme value at (*a*, *b*), we must have $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$. Using these

$$f(a+h\ ,b+k)-f(a,b)=\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b)+2hk\frac{\partial^2 f}{\partial x\partial y}(a,b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]+...$$

Let us take $A=\frac{\partial^2 f}{\partial x^2}(a,b), B=\frac{\partial^2 f}{\partial x\partial y}(a,b), C=\frac{\partial^2 f}{\partial y^2}(a,b)$. Then

$$f(a+h\ ,\ b+k)-f(a,b)=\frac{1}{2!}\left(Ah^2+hkB+Ck^2\right)+...$$
$$=\frac{1}{2!A}\left[A^2h^2+2hk\,A\,B+A\,C\,k^2\right]+...$$
$$=\frac{1}{2!A}\left[A^2h^2\ +\ 2hk\ A\ B\ +\ B^2k^2\ +\ A\,C\,k^2\ -\ B^2k^2\right]+...$$
$$=\frac{1}{2!A}\left[(Ah\ +\ Bk)^2\ +\ \left(AC-B^2\right)k^2\right]+...$$

Neglecting higher degree terms, the sign of $f(a+h,b+k)-f(a,b)$ depends on $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$. In this, $(Ah+Bk)^2$ and k^2 are positive for all h and k. The sign of $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$ depends on the signs of $AC-B^2$ and A.

The following cases are to be considered:

Case I. If $(AC-B^2)>0$, then the square bracket in the expression $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$ is positive and the sign depends an A only. When $A > 0$, then the expression $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$ is positive and hence $f(a+h,b+k)-f(a,b)>0$, which implies $f(x,y)$ has minimum value at (a, b). When $A < 0$, then the expression $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$ is negative and hence $f(a+h,b+k)-f(a,b)<0$, which implies $f(x,y)$ has maximum value at (a,b).

Case II. If $(AC-B^2)<0$, then the sign of expression $\frac{1}{2!A}\left[(A\,h+B\,k)^2+(AC-B^2)k^2\right]$ depends on small values of h and k and can have different signs for different values of h and k. Hence $f(x,y)$ has neither maximum nor minimum at (a,b). Such a point (a, b) is called saddle point.

Case III. If $AC-B^2=0$, then $f(a+h,b+k)-f(a,b)=\frac{1}{2!A}(Ah+Bk)^2$, which may vanish for values of (h, k) for which $Ah+Bk=0$. Then sign of $f(a+h,b+k)-f(a,b)$ will depend upon the next term of Taylor's expansion. This is the doubtful case and requires further investigation.

1.8.26. Conclusions. Concluding the above theorem the following procedure is used to obtain maximum and minimum of a function $f(x,y)$

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and then solve the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. Assume the points obtained are $(x_1,y_1),(x_2,y_2),\ldots$.

2. Then, find $\frac{\partial^2 f}{\partial x^2}=A$, $\frac{\partial^2 f}{\partial x\partial y}=B$, $\frac{\partial^2 f}{\partial y^2}=C$ and calculate the values of A, B, C at the point $(x_1,y_1),(x_2,y_2),\ldots$.

3. For (x_i,y_i),

(i) If $AC-B^2>0$ and $A<0$, then $f(x,y)$ has miaximum value at (x_1,y_1)

(ii) If $AC-B^2>0$ and $A>0$, then $f(x,y)$ has minimum value at $(x_1\ ,\ y_1)$

(iii) If $AC-B^2<0$, then $(x\ ,\ y)$ has neither maximum value nor minimum value at $(x_1\ ,\ y_1)$ and $(x_1\ ,\ y_1)$ is a saddle point

(iv) If $AC-B^2=0$, then the case is doubtful and we check the sign of $f(a+h\ ,\ b+k)-f(a,b)$ for small values of h and k . $f(x,y)$ has a maximum or minima according as $f(a+h,\ b+k)-f(a,b)$ is < 0 or > 0.

1.8.27. Example. Obtain extreme values for the function

$$f(x,y)=x^3+y^3-63(x+y)+12xy.$$

Solution. Let $f(x,y)=x^3+y^3-63(x+y)+12xy$.

Then, $\dfrac{\partial f}{\partial x}=3x^2-63+12y=3(x^2+4y-21)$ and $\dfrac{\partial f}{\partial x}=3x^2-63+12y=3(y^2+4x-21)$.

Then, $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ implies

$$x^2+4y-21=0$$

$$y^2+4x-21=0$$

Solving these, we get $(x-y)[x+y-4]=0$, that is, $x-y=0$ or $x+y-4=0$

When $x+y=4$, then we have x = 5, -1. Now for $x=5$, $y=-1$ and $x=-1$, $y=5$.

Thus points are $(5,-1)$ and $(-1,\ 5)$.

When $x-y=0$, that is, $x=y$. We have x = -7, 3. Now for $y=-7$, $x=-7$ and $y=3$, $x=3$.

Thus points are $(-7,-7)$ and $(3,3)$

Now, $A=\dfrac{\partial^2 f}{\partial x^2}=6x$, $B=\dfrac{\partial^2 f}{\partial x\,\partial y}=12$ and $C=\dfrac{\partial^2 f}{\partial x\,\partial y}=6y$.

1. At (5, –1), $A=30$, $B=12$, $C=-6$ and so $AC-B^2=-180-144=-324<0$.

Therefore, f has neither maximum nor minimum at (5, –1).

2. At (–1, 5), $A=-6$, $B=12$, $C=30$ and so $AC-B^2=-180-144=-324<0$.

Therefore, f has neither maximum nor minimum at (–1, 5).

3. At (–7, –7), $A=-42$, $B=12$, $C=-42$ and so $AC-B^2=1620>0$ and $A<0$.

Thus *f* has a maximum at $(-7,-7)$ and maximum value is *f*(-7,-7) = 784.

4. At $(3,3)$, $A=18$, $B=12$, $C=18$ and so $AC-B^2=324-144=180>0$ *and* $A=18>0$.

Thus *f* has a minimum at (3, 3) and minimum value is *f*(3,3) = -216.

1.8.28. Example. A rectangular box open from top is to have volume 3 cubic unit. Obtain the dimensions of the box requiring least material for its construction.

Solution. Let x, y, z be the edges of the open box and S be its surface. Since box is open, so

$$S = xy + 2yz + 2zx$$

Also it is given that $xyz = 32$

or $$z = \frac{32}{xy}$$

Therefore, we have $S = xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right) = xy + \frac{64}{x} + \frac{6y}{y}$

Then, $\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$, $\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$.

For extreme values, $\frac{\partial S}{\partial y} = 0$ and $\frac{\partial S}{\partial y} = 0$

and so $y - \frac{64}{x^2} = 0$ and $x - \frac{64}{y^2} = 0$.

Solving these, we get

$$x\left(64 - x^3\right) = 0$$

which implies, either $x = 0$ or $64 - x^3 = 0$,

that is, $x = 0$ or $x = 4$.

However, $x = 0$ is not possible as in that case *y* does not exist.

When $x = 4$ then $y = \frac{64}{16} = 4$, and so stationary point is (4, 4).

Now, $A = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $B = \frac{\partial^2 S}{\partial x \partial y} = 1$, $C = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

Then, $AC - B^2 = \frac{(128)^2}{x^3 y^3} - 1$.

Now, at (4, 4), $AC - B^2 = \frac{(128)^2}{64 \times 64} - 1 = 4 - 1 = 3 > 0$.

Also, $A = \frac{128}{4^3} = 2 > 0$

Hence S is least at (4, 4) and $z=\dfrac{32}{xy}=\dfrac{32}{4\times 4}=2$.

Therefore, dimensions of the box requiring least material for its construction are 4 ft and 2ft.

1.8.29. Lagrange's method of undetermined multipliers.

Lagrange's method of undetermined multipliers is used to find the extreme values of a function of three or more variables when the variables are not independent but have some relation between them.

Let $f(x,\ y,\ z)$ be the given function and the relation for $x,\ y,\ z$ is

$$\phi(x,\ y,\ z)=0 \tag{1}$$

At a stationary point of $f(x,\ y,\ z)$,

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0,\quad \frac{\partial f}{\partial z}=0$$

Therefore,
$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \tag{2}$$

Differentiating (1), we get

$$\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy+\frac{\partial \phi}{\partial z}dz=0 \tag{3}$$

Multiplying (3) by λ and adding to (2), we get

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial x}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz=0$$

Therefore,
$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \tag{4}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \tag{5}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \tag{6}$$

Solving (1), (4), (5) and (6), we obtain some values of $x,\ y,\ z$ for which $f(x,\ y,\ z)$ is maximum or minimum.

1.8.30. Example. Find the minimum value of the function $x^2+y^2+z^2$ subject to the condition $x+y+z=3a$

Solution. Let
$$f(x,y,z)=x^2+y^2+z^2 \tag{1}$$

and
$$\phi(x,\ y,\ z)=x+y+z-3a \tag{2}$$

Then, $f_x=2x,\ f_y=2y,\ f_z=2z$,

and $\phi_x=\ \phi_y=\ \phi_z=1$,

For stationary points, due to Lagrange's condition, we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$2x + \lambda = 0 \qquad \text{or} \qquad x = \frac{-\lambda}{2}$$

$$2y + \lambda = 0 \qquad \text{or} \qquad y = \frac{-\lambda}{2}$$

$$2z + \lambda = 0 \qquad \text{or} \qquad z = \frac{-\lambda}{2}$$

Now $\qquad x + y + z = 3a \qquad (3)$

Using values of x, y, z in (3), we get

$$\frac{-\lambda}{2} \; \frac{-\lambda}{2} \; \frac{-\lambda}{2} = 3a$$

or $$\frac{-3\lambda}{2} = 3a \qquad \text{or} \qquad \lambda = -2a$$

Therefore, $\qquad x = a, \quad y = a, \quad z = a$

Differentiating (3) partially w.r.t. x and y, we have

$$1 + 0 + \frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad 0 + 1 + \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial x} = -1 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -1$$

and from (1)

$$\frac{\partial f}{\partial x} = 2x + 0 + 2z\frac{\partial z}{\partial x} = 2x - 2z$$

$$\frac{\partial f}{\partial y} = 0 + 2y + 2z\frac{\partial z}{\partial y} = 2y - 2z$$

$$A = \frac{\partial^2 f}{\partial x^2} = 2 - 2\frac{\partial z}{\partial x} = 2 + 2 = 4$$

$$C = \frac{\partial^2 f}{\partial y^2} = 2 - \frac{2\partial z}{\partial y} = 2 + 2 = 4$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y - 2z) = -2\frac{\partial z}{\partial x} = 2$$

Thus, $\quad AC - B^2 = (4)(4) - 4 = 12 > 0$ and $A = 4 > 0$.

Hence the given function has a minimum at the point given by (3) and minimum value is $x^2 + y^2 + z^2 = a^2 + a^2 + a^2 = 3a^2$

1.8.31. Exercise.

1. Examine for maximum and minimum values of the following

(i) $x^3 - 3axy + y^3$ **(ii)** $y^2 + x^2 y + x^4$

(iii) $x^2 + y^2 + 6x + 12$ **(iv)** $x^3 + y^3 - 3x - 12y + 20$

(v) $x^2y^2 - 5x^2 - 8xy - 5y^2$ **(vi)** $xy(a - x - y)$

(vii) $f(x, y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$ $a,\ b > 0$

2. Show that $f(x, y) = (y - x)^4 + (x - 2)^4$ has minimum at (2, 2).

3. Show that $f(x, y) = (x - y)^2 + x^3 - y^3 + x^5$ has neither a maximum nor minimum at (0, 0).

4. Show

1. Verify Euler's theorem for $u = \frac{x^2 y^2}{x^2 + y^2}$.
2. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.
3. Find the dimensions of a rectangular box without a top of maximum volume whose surface area is 108 sq.cms.

1.10. Summary. In this chapter, we discussed about various aspects of calculus, like differentiability, partial derivatives, total derivatives and their applications to the maximization and minimization problems.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
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2

Integration

Structure

2.1. Introduction. This chapter contains results related to finding the integration of a given function which help students in further studies of curves in various fields.

2.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Integration.
(ii) Definite integrals.
(iii) Finding area.
(iv) Leontiff Input-Output Model.

2.1.2. Keywords. Integrate, Model, Definite Integral.

2.2. Integration. We will consider the inverse process of differentiation. In differentiation, we find the differential co-efficient of a given function while in integration if we are given the differential co-efficient of a function, we have to find the function. That is why integration is called anti-derivative i.e. in differentiation if $y = f(x)$ we find $\frac{dy}{dx}$. In integration, we are given $\frac{dy}{dx}$ and we have to find y. This integration is also called indefinite integral.

2.2.1. Definition of Integration

Integration is the inverse process of differentiation.

If $\frac{d}{dx}[\varphi(x)] = f(x)$ then

$\varphi(x)$ is called the integral or anti-derivative or primitive of $f(x)$ with respect to x.

Symbolically, it is written as

$$\int f(x)dx = \varphi(x)$$

The symbol $\int \quad dx$ denotes integration w.r.t. x. Here dx conveys that x is a variable of integration. The given function whose integral is to be found, is known as integrand.

2.2.2. Example. $\frac{d}{dx}(x^2) = 2x$

$$\therefore \int 2x \; dx = x^2.$$

2.2.3. Constant of integration

We know that $\frac{d}{dx}(x^3) = 3x^2$

Therefore integral of $3x^2$ may be $x^3, x^3 + 1$ or $x^3 + C$ where C is any arbitrary constant. Thus

$$\int 3x^2 \; dx = x^3 + C$$

2.2.4.Example. Find $\int 5x^6 \; dx$

Solution. $\int 5x^6 \; dx = 5\int x^6 \; dx = 5 \times \frac{x^7}{7} + C = \frac{5}{7}x^7 + C$

2.2.5. Standard Formulae

1. $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1 \quad \left[since \; \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n\right]$
2. $\int \frac{1}{x} dx = log_e x + C \quad \left[since \; \frac{d}{dx}(log_e x) = \frac{1}{x}\right]$
3. $\int e^x \; dx = e^x + C, \quad \left[since \; \frac{d}{dx}(e^x) = e^x\right]$
4. $\int a^x \; dx = \frac{a^x}{log_e a} + C, \quad \left[since \; \frac{d}{dx}\left(\frac{a^x}{log_e a}\right) = a^x\right]$
5. $\int e^{ax+b} \; dx = \frac{e^{ax+b}}{a} + C, \quad \left[since \; \frac{d}{dx}(e^{ax+b}) = ae^{ax+b}\right]$

6. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$, $\left[since\ \frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{a(n+1)}\right) = (ax+b)^n\right]$ $(if\ n \neq -1)$

7. $\int \frac{dx}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$ $\left[since\ \frac{d}{dx}\left(\frac{\log|ax+b|}{a}\right) = \frac{1}{ax+b}\right]$

2.2.6. Theorem. The integral of the product of a constant and a function is equal to the product of a constant, and integral of the function i.e., $\int kf(x)dx = k\int f(x)dx$, k being a constant.

Proof. Let $\int f(x)dx = \varphi(x)$, $\therefore \frac{d}{dx}[\varphi(x)] = f(x)$

Now $\frac{d}{dx}[k\varphi(x)] = k.\frac{d}{dx}[\varphi(x)]$

[Since, the derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function]

$$= kf(x) \qquad \left(as\ \frac{d}{dx}[\varphi(x)] = f(x)\right)$$

Thus, by definition

$$\int k.f(x)dx = k.\varphi(x) = k.\int f(x)dx.$$

2.2.7. Theorem. The integral of the sum or the difference of two functions is equal to the sum or difference of their integrals i.e., $\int [f_1(x) \pm f_2(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx$.

Proof. Let $\int f_1(x)dx = \varphi_1(x)$ and $\int f_2(x)dx = \varphi_2(x)$

Therefore,

$$\frac{d}{dx}[\varphi_1(x)] = f_1(x) \text{ and } \frac{d}{dx}[\varphi_2(x)] = f_2(x)$$

Now $\frac{d}{dx}[\varphi_1(x) \pm \varphi_2(x)] = \frac{d}{dx}[\varphi_1(x)] \pm \frac{d}{dx}[\varphi_2(x)] = f_1(x) \pm f_2(x)$

[Since, the derivative of the sum or difference of two functions is equal to the sum or difference of their derivatives].

Therefore, by definition of the integral of a function

$$\int [f_1(x) \pm f_2(x)]dx = \varphi_1(x) \pm \varphi_2(x) = \int f_1(x)dx \pm \int f_2(x)dx.$$

Remark. We can extend this theorem to a finite number of functions and can have the following result.

$$\int [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]dx = \int f_1(x)dx \pm \int f_2(x)dx \pm \cdots \pm \int f_n(x)dx.$$

2.2.8. Example. Write down the integral of

(i) x^2 (ii) x^{-9} (iii) 1

(iv) $\sqrt{x}$ (v) $\frac{1}{x^2}$ (vi) $x^{-2/3}$

Solution.

(i) $\int x^2dx = \frac{x^{2+1}}{2+1} + C = \frac{1}{3}x^3 + C$

(ii) $\int x^{-9}dx = \frac{x^{-9+1}}{-9+1} + C = \frac{1}{-8}x^{-8} + C = -\frac{1}{8x^8} + C$

(iii) $\int 1dx = \int x^0 dx = \frac{x^{0+1}}{0+1} + C = x + C$

(iv) $\int x^{1/2}dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{2}{3}x^{3/2} + C$

(v) $\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} + C = -\frac{1}{x} + C$

(vi) $\int x^{-2/3}dx = \frac{x^{-2/3+1}}{-2/3+1} + C = 3x^{1/3} + C.$

2.2.9. Example. Find the integrals of the following

(i) $\sqrt{x} - \frac{1}{\sqrt{x}}$ (ii) $\frac{(1+x)^2}{x^3}$ (iii) $\frac{x^4}{x^2+1}$

(iv) $x\sqrt{x+2}$ (v) $(1+x)\sqrt{1-x}$

Solution.

(i) $\int \sqrt{x} - \frac{1}{\sqrt{x}}\, dx = \int \left(x^{1/2} - x^{-1/2}\right)dx$

$$= \frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2} = \frac{2x^{3/2}}{3} - 2x^{1/2} + C$$

(ii) $\int \frac{(1+x)^2}{x^3}\, dx = \int \left(\frac{1+2x+x^2}{x^3}\right) dx = \int \left(\frac{1}{x^3} + \frac{2}{x^2} + \frac{1}{x}\right) dx$

$$= \int x^{-3}\, dx + 2 \int x^{-2}\, dx + \int \frac{1}{x} dx$$

$$= \frac{x^{-2}}{-2} + 2\frac{x^{-1}}{-1} + logx + C$$

$$= -\frac{1}{2x^2} - \frac{2}{x} + \log x + C.$$

(iii) $\int \frac{x^4}{x^2+1}\, dx = \int \left(\frac{(x^4-1)+1}{x^2+1}\right) dx$

$$= \int \frac{x^4-1}{x^2+1} dx + \int \frac{1}{x^2+1} dx = \int (x^2-1)\, dx + \int \frac{1}{x^2+1} dx$$

$$= \frac{x^3}{3} - x + \tan^{-1} x + C$$

(iv) $I = \int x\sqrt{x+2}\, dx$

$$= \int [(x+2) - 2]\sqrt{x+2}\, dx$$

$$= \int (x+2)\sqrt{x+2}\; dx - \int 2\sqrt{x+2}\, dx$$

$$= \int (x+2)^{3/2}\, dx - 2\int (x+2)^{1/2}\, dx$$

$$= \frac{(x+2)^{5/2}}{5/2} - 2\frac{(x+2)^{3/2}}{3/2} + C$$

$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$

(v) I $= \int (1+x)\sqrt{1-x}\, dx$

$= \int [2-(1-x)]\sqrt{1-x}\, dx$

$= 2\int (1-x)^{1/2}\, dx - \int (1-x)^{3/2}\, dx$

$= \frac{2(1-x)^{3/2}}{-3/2} - \frac{(1-x)^{5/2}}{-5/2} + C$

$= -\frac{4}{5}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + C.$

2.2.10. Example. Integrate $a^{3x+3}\, dx,\ a \neq -1$

Solution. I $= \int a^{3x+3} dx = \int a^{3x}. a^3 dx$

$= a^3 \int a^{3x} dx$

$= a^3 \int e^{3x \log a} dx$ (Since $e^{\log f(x)} = f(x)$

Therefore, $e^{\log a^{3x}} = a^{3x}$

Also $e^{\log\ a^{3x}} = e^{3xlog\ a}$

Therefore, $a^{3x} = e^{3xlog\ a}$

$= a^3 \int e^{(3\log a)x} dx$

$= a^3 \frac{e^{(3\log\ a)x}}{3\log a} + C$

$= a^3 \frac{e^{3x\log\ a}}{3\log a} + C = \frac{a^3 a^{3x}}{3\log a} + C = \frac{a^{3x+3}}{3\log a} + C.$

2.3. Integration by Substitution.

By substitution, many functions can be converted into smaller functions which can be integrated easily.

When we apply method of substitution for finding the value of $\int f(x)\ dx$ and if $x = f(t)$ where t is a new variable then $f(x)$ is converted into $F[f(t)]$ and also dy/dx.

Now $x = f(t)$

Therefore, $\frac{dx}{dt} = f'(t)$ or $dx = f'(t)\, dt$.

Two important forms of integrals :

(i) $\int \frac{f'(x)}{f(x)} dx = log|f(x)| + C$

(ii) $\int [f(x)]^n . f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1}$ $when\ n \neq -1$.

2.3.1. Example. Evaluate the following :

(i) $\int \frac{2x+9}{x^2+9x+10} dx$ (ii) $\int \frac{6x-8}{3x^2-8x+5} dx$

(iii) $\int 3x^2 . e^{x^3} dx$ (iv) $\int \frac{e^{1/x^2}}{x^3} dx$

(v) $\int \frac{\log x}{x} dx$ (vi) $\int \frac{1}{x \log_e x} dx$

(vii) $\int \frac{x^3}{\sqrt{1+x^3}} dx$ (viii) $\int \frac{1}{x+\sqrt{x}} dx$

Solution.

(i) $\mathrm{I} = \int \frac{2x+9}{x^2+9x+10} dx$

Put $x^2 + 9x + 10 = t$

Therefore, $2x + 9 = \frac{dx}{dt}$ or $(2x + 9)dx = dt$

Therefore, $\mathrm{I} = \int \frac{dt}{t} = \log|t| + C = \log|x^2 + 9x + 10| + C$

(ii) $\mathrm{I} = \int \frac{6x-8}{3x^2-8x+5} dx$

Put $3x^2 - 8x + 5 = t$

Therefore, $6x - 8 = \frac{dx}{dt}$ or $(6x - 8)dx = dt$

Therefore, $\mathrm{I} = \int \frac{dt}{t} = \log|t| + C = \log|3x^2 - 8x + 5| + C.$

(iii) $\mathrm{I} = \int 3x^2 . e^{x^3} dx$

Put $x^3 = t$

Therefore, $3x^2 = \frac{dx}{dt}$ or $3x^2 dx = dt$

Therefore, $\mathrm{I} = \int 3x^2 . e^{x^3} dx = \int e^t \, dt = e^t + C = e^{x^3} + C.$

(iv) Let $\frac{1}{x^2} = t$ or $x^{-2} = t$

Therefore, $-\frac{2}{x^3} dx = dt$ or $\frac{1}{x^3} dx = -\frac{1}{2} dt$

Thus, $\int \frac{e^{1/x^2}}{x^3} dx = \int e^{1/x^2} \frac{1}{x^3} dx = \int e^t \left(-\frac{1}{2}\right) dt$

$$= -\frac{1}{2} \int e^t \, dt = -\frac{1}{2} e^t + C = -\frac{1}{2} e^{1/x^2} + C.$$

(v) Let $\log x = t.$ So $\frac{1}{x} dx = dt$

Therefore,

$$\int \frac{\log x}{x} dx = \int \log x . \frac{1}{x} \, dx = \int t \, dt = \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C.$$

(vi) Let $\log_e x = t$, so $\frac{1}{x}dx = dt$

Therefore,

$$\int \frac{1}{x\log_e x}dx = \int \frac{1}{\log_e x}.\frac{1}{x}\,dx = \int \frac{1}{t}dt = \log_e t + C = \log_e(\log_e x) + C.$$

(vii) Let $1 + x^3 = t^2$ or $x^2dx = \frac{2}{3}t\,dt$

Therefore, $\int \frac{x^3}{\sqrt{(1+x^3)}}dx = \int \frac{x^3}{\sqrt{(1+x^3)}}.x^2dx = \int \frac{t^2-1}{t}.\frac{2}{3}t\,dt$

$$= \left(\frac{2}{3}\right)\int (t^2-1)dt = \frac{2}{3}\left(\frac{1}{3}t^3 - t\right) + C$$

$$= \frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + C$$

(viii) $\int \frac{1}{x+\sqrt{x}}dx = \int \frac{1}{\sqrt{x}(\sqrt{x}+1)}dx$

Let $\sqrt{x} = t$, so $\frac{1}{2\sqrt{x}}dx = dt$ or $\frac{1}{\sqrt{x}}dx = 2dt$

Therefore, $\int \frac{1}{x+\sqrt{x}}dx = 2\int \frac{7}{\sqrt{x}(\sqrt{x}+1)} = \int \frac{1}{t+1}dx$

$$= 2\log(t+1) + C = 2\log\left(\sqrt{x}+1\right) + C.$$

2.3.2. Example. Integrate the following :

(i) $x\sqrt{x+2}$ (ii) $\frac{2+3x}{3+2x}$ (iii) $\frac{(x+1)(x+\log x)^2}{x}$ (iv) $\frac{1}{e^{x-1}}$

Solution.

(i) $I = \int x\sqrt{x+2}\,dx$

Putting $x + 2 = t$ implies $x = t - 2$

Therefore, $dx = dt$

$$I = \int (t-2)t^{1/2}\,dt = \int t^{3/2}\,dt - 2\int t^{1/2}\,dt$$

$$= \frac{t^{5/2}}{5/2} - 2\frac{t^{3/2}}{3/2} + C = \frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2} + C$$

$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$$

(ii) $I = \int \frac{2+3x}{3+2x}\,dx$

Putting $3 - 2x = t$ implies $x = \frac{3-t}{2}$

Therefore, $dt = -2dx$ implies $dx = -\frac{dt}{2}$

$$I = -\frac{1}{2}\int \frac{2+3\left(\frac{3-t}{2}\right)}{t}dt = -\frac{1}{2}\int \frac{2+\frac{9}{2}-\frac{3}{2}t}{t}\frac{dt}{2}$$

$$= -\int \frac{dt}{t} - \frac{9}{4}\int \frac{dt}{t} + \frac{3}{4}\int dt$$

$$= -\log|t| - \frac{9}{4}\log|t| + \frac{4}{3}t + C$$

$$= -\log|3-2x| - \frac{9}{4}\log|3-2x| + \frac{3}{4}(3-2x) + C$$

$$= \frac{3}{4}(3-2x) - \log|3-2x| - \frac{9}{4}\log|3-2x| + C$$

(iii) $\mathrm{I} = \int \frac{(x+1)(x+\log x)^2}{x} dx$

Put $x + \log x = t,$ therefore $\left(1+\frac{1}{x}\right)dx = dt$ or $\left(\frac{x+1}{x}\right)dx = dt$

Thus $I = \int (x+\log x)^2 \left(\frac{x+1}{x}\right) dx = \int t^2 dt = \frac{t^3}{3} + C = \frac{1}{3}\int (x+\log x)^3 + C.$

(iv) $\int \frac{1}{e^x - 1} dx = \int \frac{dt}{t(t-1)} = \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$

$$= \log(t-1) - \ logt + C$$

$$= log\left(\frac{t-1}{t}\right) = log\left(\frac{e^x - 1}{e^x}\right) + C$$

2.4. Integral of the product of two functions.

If u and v be two functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

implies $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Integrating both sides w.r.t x, we get

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \text{.... (1)}$$

Let $u = f_1(x)$ and $\frac{dv}{dx} = f_2(x)$

Since $\frac{dv}{dx} = f_2(x),$ therefore $\int f_2(x)\,dx = v$

Hence (1) becomes

$$\int f_1(x)f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int [f_1^{'}(x)\int f_2(x)dx]\,dx.$$

In words, this rule of integration by parts can be stated as :

Integral of the product of two functions

= First function . Integral of the second

-Integral of [diff. coeff. of the first . Integral of the second)

Integral of the product of two functions

In finding integrals by this method proper choice of 1st and 2nd function is essential. Although there is no fixed law for taking 1st and 2nd function and their choice is possible by practice, yet following rule is helpful in the choice of functions 1st and 2nd.

(i) If the two functions are of different types take that function as Ist which comes first in the word **ILATE.**

Where I, stands for Inverse circular function.

L, stands for Logarithmic function.

A, stands for Algebraic function.

T, stands for Trigonometrical function.

and E, stands for Exponential function.

(ii) If both the functions are trigonometrical take that function as 2nd whose integral is simpler.

(iii) If both the functions are algebraic take that function as 1st whose d.c. is simpler.

(iv) Unity may be taken as one of the functions.

(v) The formula of integration by parts can be applied more than once if necessary.

2.4.1. Example. Evaluate $\int x^n \log x \, dx$

Solution. Let $\mathrm{I} = \int x^n \log x \, dx = \int (\log x) x^n \, dx$

So
$$\mathrm{I} = (\log x)\frac{x^{n+1}}{n+1} - \int \frac{1}{x}\frac{x^{n+1}}{n+1} dx$$
$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\int x^n \, dx$$
$$= \frac{x^{n+1}(\log x)}{n+1} - \frac{1}{n+1}\frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

2.4.2. Example. Evaluate $\int xe^x \, dx$

Solution. Let $\mathrm{I} = \int xe^x \, dx$

[Here x is algebraic function and e^x is exponential function and A occurs before T in ILATE, therefore, we take x as 1st and e^x as 2nd functions].

$$I = \int xe^x \, dx = x \int e^x \, dx - \int \left(\frac{d}{dx}(x) \int e^x \, dx\right) dx$$
$$= xe^x - \int 1.e^x \, dx = xe^x - e^x + C = e^x(x-1) + C.$$

2.4.3. Example. Evaluate $\int x^3 e^{-x} \, dx$

Solution. Let $\mathrm{I} = \int x^3 e^{-x} \, dx = x^3(-e^{-x}) - \int 3x^2(-e^{-x}) dx$
$$= -x^3 e^{-x} + 3\int x^2 e^{-x} dx$$
$$= -x^3 e^{-x} + 3[x^2(-e^{-x}) - \int 2x(-e^{-x}) dx]$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6\int xe^{-x}dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} + 6[x(-e^{-x}) - \int 1(-e^{-x})dx$$

$$= -x^3e^{-x} - 3x^2e^{-x} - 6e^{-x} + 6xe^{-x} + C$$

$$= -e^{-x}(x^3 + 3x^2 - 6x + 6) + C$$

2.4.4. Example. Integrate $x^3e^{x^2}$

Solution. I = $\int x^3e^{x^2}\,dx$

Put $x^2 = t$, therefore, $2xdx = dt$ or $xdx = \frac{dt}{2}$

$$\text{I} = \int x^3e^{x^2}\,dx = \int x^2e^{x^2}x\,dx = \int te^t\frac{dt}{2} = \frac{1}{2}\int te^t\,dt$$

$$= \frac{1}{2}[te^t - \int 1.e^t\,dt]$$

$$= \frac{1}{2}te^t - \frac{1}{2}e^t + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

2.4.5. Example. Evaluate $\int x^2e^{ax}\,dx$

Solution. Let $I = \int x^2e^{ax}\,dx$

$$= x^2\left(\frac{e^{ax}}{a}\right) - \int 2x\frac{e^{ax}}{a}dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[x\left(\frac{e^{ax}}{a}\right) - \int 1.\frac{e^{ax}}{a}dx\right]$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[x\frac{e^{ax}}{a} - \frac{1}{a}e^{ax}\right] + C$$

$$= e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^2}\right) + C.$$

2.4.6. Example. Evaluate$\int \log x\,dx$

Solution. Let I= $\int \log x\,dx = \int(\log x).1\,dx$

Integrating by parts, taking $log\ x$ as the 1st function

$$= \log x\,(x) - \int\frac{1}{x}.x\,dx = xlog\ x - \int 1\,dx$$

$$= x\log x - x + C = x(\log x - 1) + C.$$

2.4.7. Example. Evaluate $\int(\log x)^2.\,x\,dx$

Solution. Let $I = \int(\log x)^2.\,x\,dx$

$$= (\log x)^2.\frac{x^2}{2} - \int(2\log x).\frac{1}{x}.\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}(\log x)^2 - \int(\log x).\,x\,dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left(\log x \frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\,dx\right)$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\left[(\log x)^2 - \log x + \frac{1}{2}\right] + C$$

2.4.8. Example. Evaluate $\int e^x(1+x)\log(xe^x)\,dx$

Solution. Let I= $\int e^x(1+x)\log(xe^x)\,dx$

Put $xe^x = t$, therefore, $e^x(1+x)dx = dt$

Therefore, $I = \int(\log t).1\,dt$

$$= \log t.(t) - \int \frac{1}{t}.t\,dt$$

$$= t\log t - \int 1.dt = t\log t - t + C$$

$$= t(\log t - 1) + C = (xe^x)[\log(xe^x) - \log e] + C$$

$$= (xe^x)\log\left(\frac{xe^x}{e}\right) + C$$

2.4.9. Example. Evaluate $\int \frac{\log x}{(x+1)^2}\,dx$

Solution. Let I= $\int \log x.\frac{1}{(x+1)^2}\,dx$

Now integrating by parts, taking $log\ x$ as first function

$$\mathrm{I} = \log x.\frac{-1}{1+x} - \int \frac{1}{x}.\frac{-1}{1+x}\,dx = -\frac{\log x}{1+x}\,dx = -\frac{\log x}{1+x} + \int \frac{1}{x(1+x)}\,dx$$

$$= -\frac{\log x}{1+x} + \int\left(\frac{1}{x} + \frac{1}{1+x}\right)dx$$

$$= -\frac{\log x}{1+x} + \log|x| - \log|1+x| + C$$

$$= -\frac{\log x}{1+x} + \log\left|\frac{x}{1+x}\right| + C.$$

2.5. Integration by partial fractions.

2.5.1. Rational Function. An expression of the form $\frac{f(x)}{\varphi(x)}$ where $f(x)$ and $\varphi(x)$ are rational integral algebraic functions or polynomials.

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$$

$$\varphi(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n.$$

Where m, n are positive integers and $a_0, a_1, a_2, \ldots, a_m, b_0, b_1, b_2, \ldots, b_n$ are constants is called a rational function or rational fraction. It is assumed that $f(x)$ and $\varphi(x)$ have no common factor.

e.g. $\frac{x+1}{x^3+x^2-6x}, \frac{x-1}{(x+1)(x^2+1)}$ are rational functions.

Such fractions can always be integrated by splitting the given fraction into partial fractions.

Note on Partial Fractions

1. **Proper rational algebraic fraction.** A proper rational algebraic fraction is a rational algebraic fraction in which the degree of the numerator is less than that of the denominator.

2. The degree of the numerator $f(x)$ must be less than the degree of denominator $\varphi(x)$ and if the degree of the numerator of a rational algebraic fraction is equal to or greater than, that of the denominator, we can divide the numerator by the denominator until the degree of the remainder is less than that of the denominator.

   Then

   Given fraction = a polynomial + a proper rational algebraic fraction.

   For example, consider a rational algebraic fraction.

   $$\frac{x^2}{(x-1)(x-2)} = \frac{x^2}{x^2-3x+2}$$

Hence the degree of the numerator is 3 and the degree of the denominator is 2. We divide numerator by denominator.

Therefore, $\frac{x^2}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)}$

Working rule.

(i) The degree of the numerator (x) must be less than the degree of denominator $\varphi(x)$ and if not so, then divide $f(x)$ by $\varphi(x)$ till the remainder of a lower degree than $\varphi(x)$.

(ii) Now break the denominator $\varphi(x)$ into linear and quadratic factors.

(iii) (a) Corresponding to non-repeated linear factor of $(x - a)$ type in the denominator $\varphi(x)$.. Put a partial fraction of the form $\frac{A}{x-\alpha}$.

Therefore, the partial fraction of $\frac{x^2}{(x+2)(x-4)(x-5)}$ are of the form $\frac{A}{x+2} + \frac{B}{x-4} + \frac{C}{x-5}$

(b) Corresponding to non-repeated quadratic factor $(ax^2 + bx + c)$ of $\varphi(x)$, partial fraction will be of the form $\frac{Ax+b}{ax^2 + bx + c}$

For example, the partial fraction of

$$\frac{2x-3}{(x-1)(x-4)^2(x^2-5x+10)} = \frac{A}{(x-1)} + \frac{B}{x-4} + \frac{C}{(x-4)^2} + \frac{D}{x^2-5x+10}$$

(c) Corresponding to a repeated quadratic factor of the form $(ax^2 + b + c)^m$ in $\varphi(x)$, there corresponds m partial fractions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

Therefore the partial fractions of

$$\frac{3x-5}{(x+5)(x^2+7x+8)^2} = \frac{A}{x+5} + \frac{Bx+C}{x^2+7x+8} + \frac{Dx+E}{(x^2+7x+8)^2}$$

Thus we see that when we resolve the denominator $\varphi(x)$ into real factors, they can be of four types :

(a) Linear non-repeated.

(b) Linear repeated.

(c) Quadratic non-repeated.

(d) Quadratic repeated.

The proper fraction $\frac{f(x)}{\varphi(x)}$ is equal to the sum of partial fractions as suggested above. After this, multiply both sides by $\varphi(x)$. The relation, we get will be an identity. So the values of the constants of R.H.S. will be obtained by equating the coefficients of like powers of x, and then

solving the equation so obtained. Sometimes we can get the values of constants by some short cut methods i.e., by giving certain values to x etc.

2.5.2. Example. Evaluate the following

(i) $\int \frac{3x+2}{(x-2)(2x+3)} dx$ (ii) $\int \frac{3x-1}{(2x+1)(3x+2)(6x-1)} dx$

Solution. (i) Let $\frac{3x+2}{(x-2)(2x+3)} = \frac{A}{x-2} + \frac{B}{2x+3}$

Multiplying both sides by $(x-2)\,(2x+3)$

$$3x + 2 = A(2x+3) + B(x-2)$$

Put $x = -\frac{3}{2}$, we have $B = \frac{5}{7}$

Put $x = 2$, we have $B = \frac{8}{7}$

Therefore, $\frac{3x+2}{(x-2)(2x+3)} = \frac{8}{7(x-2)} + \frac{5}{7(2x+3)}$

Thus, $\int \frac{3x+2}{(x-2)(2x+3)} dx = \int \frac{8}{7(x-2)} dx + \int \frac{5}{7(2x+3)} dx$

$$= \frac{8}{7}\log|x-2| + \frac{5}{7}\log|2x+3| + C$$

(ii) Let $\frac{3x-1}{(2x+1)(3x+2)(6x-1)} = \frac{A}{2x+1} + \frac{B}{3x+2} + \frac{C}{6x-5}$

Multiplying both sides by $(2x+1)\,(3x+2)\,(6x-5)$

$$(3x-1) = A(3x+2)(6x-5) + B(2x+1)(6x-5) + C(2x+1)(3x+2)$$

Put $x = -\frac{1}{2}$, we have $A = \frac{5}{8}$

Put $x = -\frac{2}{3}$, we have $B = -5$

Put $x = \frac{5}{6}$, we have $C = \frac{1}{8}$

Therefore, $\frac{3x-1}{(2x+1)(3x+2)(6x-1)} = \frac{5}{8(2x+1)} - \frac{1}{3x+2} + \frac{1}{8(6x-5)}$

Thus, $\int \frac{3x-1}{(2x+1)(3x+2)(6x-1)} dx = \int \frac{5}{8(2x+1)} dx - \int \frac{1}{3x+2} dx + \int \frac{1}{8(6x-5)} dx$

$$= \frac{5}{16} \log|2x+1| - \frac{1}{3} \log|3x+2| + \frac{1}{48} \log|6x-5| + C$$

2.5.3. Example. Evaluate

(i) $\int \frac{17x-2}{4x^2+7x-2} dx$ (ii) $\int \frac{dx}{x-x^3}$

Solution. (i) $\frac{17x-2}{4x^2+7x-2} = \frac{17x-2}{(x+2)(4x-1)} = \frac{A}{x+2} + \frac{B}{4x-1}$

$$17x - 2 = A(4x-1) + B(x+2)$$

Put $x = \frac{1}{4}$, we have $B = 1$

Put $x = -2$, we have $A = 4$

Therefore, $\frac{17x-2}{4x^2+7x-2} = \frac{4}{x+2} + \frac{1}{4x-1}$

Thus, $\int \frac{17x-2}{4x^2+7x-2} dx = \int \frac{4}{x+2} dx + \int \frac{1}{4x-1} dx$

$$= 4 \log|x+2| + \frac{1}{4} \log|4x-1| + C.$$

(ii) $\int \frac{dx}{x-x^3} = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1-x)(x+x)}$

Let $\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$

Multiplying both sides by $x(1-x)(1+x)$, we get

$$1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$$

Putting $x = 0, 1$ and -1, we get $A = 1$, $B = \frac{1}{2}$, $C = -\frac{1}{2}$

Putting these values of A, B and C, we get

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

Then,

$$\int \frac{dx}{x-x^2} = \int \left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right] dx$$

$$= \log|x| - \frac{1}{2} \log|1-x| - \frac{1}{2} \log|1+x| + C$$

$$= \frac{1}{2}[2\log|x| - \log|1-x| - \log|1+x|] + C$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$$

2.5.4. Example. Evaluate (i) $\int \frac{dx}{1+3e^x+2e^{2x}}$ (ii) $\int \frac{dx}{6(\log x)^2+7\log x+2}$

Solution. (i) Put $e^x = t$, therefore $e^x dx = dt$

$$I = \int \frac{dt}{e^x(1+3t+2t^2)} = \int \frac{dt}{t(2t+1)(t+1)}$$

Now $$\frac{1}{t(2t+1)(t+1)} = \frac{1}{t} + \frac{1}{1+t} - \frac{4}{2t+1}$$

$$I = \int \frac{1}{t} dt + \int \frac{1}{1+t} dt - \int \frac{4}{2t+1} dt$$

$$= \log|t| + \log|1+t| - 2\log|2t+1| + C$$

$$= \log|e^x| + \log|e^x+1| - 2\log|2e^x+1| + C$$

$$= x + +\log|e^x+1| - 2\log|2e^x+1| + C$$

(ii) $\int \frac{dx}{6(\log x)^2+7\log x+2}$

Put $\log x = t$, then $\frac{1}{x}dx = dt$

$$I = \int \frac{dt}{6t^2+7t+2} = \int \frac{dt}{(2t+1)(3t+2)} = 2\int \frac{dt}{2t+1} - 3\int \frac{dt}{3t+2}$$

$$= \log|2t+1| - \frac{3}{2}\log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C = \log\left|\frac{2\log x+1}{3\log x+2}\right| + C$$

2.6. Definite Integral and Area.

Sometimes, in geometry and other branches of integral calculus, it becomes necessary to find the differences in two values (say a and b) of a variable x for integral values of function $f(x)$. This difference is called definite integral of $f(x)$ within limits a and b or b and a.

This definite integral is shown as follows :

$$\int_a^b f(x)\, dx$$

and is read as integration of $f(x)$ between limits a and b. As we know that if $\int f(x)\, dx = F(x)$

So $$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a),$$

where a and b are called lower and upper limits.

General Properties of Definite Integral

Property 1. $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

Property 2. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Property 4. $\int_0^a f(x)\,dx = \int_0^b f(a-x)\,dx$

Property 5. $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function of x

$$= 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ is an even function of } x$$

Note. (i) $f(x)$ is called odd function if $f(-x) = -f(x)$

(ii) $f(x)$ is called even function if $f(-x) = f(x)$

Property 6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

2.6.1. Example. Find the values of

(i) $\int_0^1 x^2\,dx$ (ii) $\int_{-1}^2 (3x-1)(2x+1)\,dx$ (iii) $\int_2^3 \frac{dx}{x^2-1}$

(iv) $\int_0^2 \frac{e^x - e^{-x}}{5}\,dx$ (v) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$ (vi) $\int_0^1 \frac{dx}{[(ax+b)(1-x)]^2}$

Solution. (i) $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

(ii) $\int_{-1}^2 (3x-1)(2x+1)\,dx = \int_{-1}^2 (6x^2+x-1)\,dx$

$$= 6\left[\frac{x^3}{3}\right]_{-1}^2 + \left[\frac{x^2}{2}\right]_{-1}^2 + [x]_{-1}^2 = 16\frac{1}{2}$$

(iii) $\int_2^3 \frac{dx}{x^2-1} = \int_2^3 \frac{dx}{x^2-1^2}$

$$= \left[\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\left[\log\frac{2}{4} - \log\frac{1}{3}\right] = \frac{1}{2}\log\frac{3}{2}$$

(iv) $\int_0^2 \frac{e^x - e^{-x}}{5}\,dx = \frac{1}{5}\int_0^2 e^x - e^{-x}\,dx$

$$= \frac{1}{5}[e^x + e^{-x}]_0^2 = \frac{1}{5}\left(e - \frac{1}{e}\right)^2$$

(v) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int_0^1 \frac{\sqrt{x+1}-\sqrt{x}}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}\,dx = \int_0^1 (\sqrt{x+1}-\sqrt{x})\,dx$

$$= \frac{2}{3}\left|(x+1)^{3/2} - x^{3/2}\right|_0^1 = \frac{4}{3}(\sqrt{2}-1).$$

(vi) $\int_0^1 \frac{dx}{[ax+b(1-x)]^2} = \int_0^1 \frac{dx}{[(a-b)x+b]^2} = \int_0^1 [(a-b)x+b]^{-2}\,dx$

$$= \left[\frac{[(a-b)x+b]^{-1}}{b-a}\right]_0^1 = \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{ab}$$

2.6.2. Example. If $\int_0^a 3x^2\,dx = 8$, find the value of a.

Solution. $\int_0^a 3x^2\,dx = 3\int_0^a x^2\,dx = 3\left[\frac{x^3}{3}\right]_0^a = a^3$

Since $\int_0^a 3x^2\,dx = 8$

Implies $a^3 = 8$ i.e. $a = 2$.

2.6.3. Example. Show that when $f(x)$ is of the form $a + bx + cx^2$, then

$$\int_0^1 f(x)\,dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

Solution. $f(x) = a + bx + cx^2$

$$f(0) = a, \quad f\left(\frac{1}{2}\right) = a + \frac{1}{2}b + \frac{1}{4}c$$

$$f(1) = a + b + c$$

RHS = $\frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = a + \frac{b}{2} + \frac{c}{3}$

LHS = $\int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx = \left|ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right|_0^1 = a + \frac{b}{2} + \frac{c}{3}$

Hence. LHS = RHS.

2.6.4. Example. Evaluate the following definite integrals

(i) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$ (ii) $\int_1^2 3x\sqrt{5-x^2}\,dx$ (iii) $\int_4^8 x\sqrt[3]{x-4}\,dx$

(iv) $\int_a^b \frac{\log x}{x}\,dx$ (v) $\int_0^2 \frac{dx}{4+x-x^2}$ (vi) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Solution. (i) $I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$

Put $1 - x^2 = t$, then $-2xdx = dt$ or $xdx = -\frac{1}{2}dt$

When $x = 0, t = 1$

When $x = \frac{1}{2},\ t = \frac{3}{4}$

Therefore, $I = \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} t^{-1/2}\,dt = -\frac{1}{2}\left[\frac{t^{1/2}}{1/2}\right]_1^{3/4} = 1 - \frac{\sqrt{3}}{2}$

(ii) $I = \int_1^2 3x\sqrt{5-x^2}\,dx$

Put $5 - x^2 = t$, therefore $-2xdx = dt$ or $xdx = -\frac{1}{2}dt$

When $x = 1, t = 5 - 1 = 4$

When $x = 2,\ t = 5 - 4 = 1$

Therefore, $I = \int_1^2 3x\sqrt{5-x^2}\,dx = -\frac{3}{2}\int_4^1 t^{\frac{1}{2}}\,dt = -\frac{3}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^1$

$$= -(1-4^{\frac{3}{2}}) = -(1-8) = 7.$$

(iii) $I = \int_4^8 x\sqrt[3]{x-4}\,dx = \int_4^8 x(x-4)^{1/3}\ dx$

Put $x-4=t$, therefore, $dx = dt$

When $x=4,\ t=0$

When $x=8,\ t=4$

Therefore, $I = \int_0^4 (t+4)(t)^{1/3}\ dx = \int_0^4 t^{4/3}\,dt + 4\int_0^4 t^{1/3}\,dt$

$$= \frac{3}{7}\left|t^{7/3}\right|_0^4 + 4\times\frac{3}{4}\left|t^{4/3}\right|_0^4$$

$$= \frac{3}{7}(4)^{7/3} + 3(4)^{4/3} = \frac{132}{7}(4)^{1/3}.$$

(iv) $I = \int_a^b \frac{\log x}{x}\,dx$

Put $\log x = t$, therefore $\frac{1}{x}dx = dt$

When $x=a,\quad t=\log a$

When $x=b,\quad t=\log b$

Therefore, $I = \int_{\log a}^{\log b} t\,dt = \left|\frac{t^2}{2}\right|_{\log a}^{\log b} = \frac{1}{2}[(\log b)^2 - (\log a)^2]$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a) = \frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$$

(v) $I = \int_0^2 \frac{dx}{4+x-x^2} = \int_0^2 \frac{dx}{4-(x^2-x)}$

$$= \int_0^2 \frac{dx}{4-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}} = \int_0^2 \frac{dx}{\frac{17}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\sqrt{\frac{17}{2}}}\left[\log\left(\frac{x-\frac{1}{2}+\sqrt{\frac{17}{2}}}{-x+\frac{1}{2}+\sqrt{\frac{17}{2}}}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left\{\log\left(\frac{\sqrt{17}+3}{\sqrt{17}-3}\right) - \log\left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)\right\}$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{4(5+\sqrt{17})}{4(5-\sqrt{17})}\right) = \frac{1}{\sqrt{17}} \log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}} \times \frac{5+\sqrt{17}}{5+\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{21+5\sqrt{17}}{4}\right)$$

(vi) $I = \int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

Put $x + 1 = t^2$, then $dx = 2tdt$

When $x = 8$, $t^2 = 9$, implies $t = 3$

When $x = 15$, $t^2 = 16$, implies $t = 4$

Therefore, $I = 2\int_3^4 \frac{dt}{t^2-2^2} = 2.\frac{1}{4}\left[\log\left|\frac{t-2}{t+2}\right|_3^4\right]$

$$= \frac{1}{2}\left(\log\left|\frac{2}{6}\right| - \log\frac{1}{5}\right) = \frac{1}{2}\left(\log\frac{1}{3} - \log\frac{1}{5}\right)$$

$$= \frac{1}{2}\log\frac{5}{3}$$

2.6.5. Example. Evaluate the following

(i) $\int_0^1 x^2 e^{2x}\, dx$ (ii) $\int_0^1 (x-2)(2x+3)e^x\, dx$

(iii) $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}} dx$ (iv) $\int_1^e \frac{e^x}{x}(1 + x\log x)\mathrm{dx}$

Solution. (i) $I = \int_0^1 x^2 e^{2x}\, dx$

$$= \left|x^2\left(\frac{e^{2x}}{2}\right)\right|_0^1 - \int_0^1 2x\left(\frac{1}{2}e^{2x}\right) dx$$

$$= \frac{1}{2}(e^2 - 0) - \left[\left|\left(\frac{xe^{2x}}{2}\right)\right|_0^1 - \int_0^1 \left(\frac{1}{2}e^{2x}\right) dx\right]$$

$$= \frac{1}{2}e^2 - \left[\frac{1}{2}e^2 - \frac{1}{2}\int_0^1 e^{2x}\, dx\right] = \frac{1}{2}\int_0^1 e^{2x}\, dx$$

$$= \frac{1}{2}\left|\frac{1}{2}e^{2x}\right|_0^1 = \frac{1}{2}(e^2 - 1).$$

(ii) $I = \int_0^1 (x-2)(2x+3)e^x\, dx$

$$= \int_0^1 (2x^2 - x - 6)e^x\, dx$$

Integrating by parts, we get

$$= [(2x^2 - x - 6)e^x]_0^1 - \int_0^1 (4x-1)e^x\, dx$$

$$= (2 - 1 - 6)e - (-6) - \int_0^1 (4x-1)e^x\, dx$$

$= -5e + 6 - \left[|(4x-1)e^x|_0^1 - \int_0^1 4e^x\,dx\right]$

$= -5e + 6 - [(4-1)e - (-1) - 4|e^x|_0^1]$

$= -5e + 6[3e + 1 - 4(e-1)]$

$= 1 - 4e.$

(iii) $I = \int_2^4 \frac{x^2+x}{\sqrt{2x+1}}dx$

Integrating by parts taking $x^2 + x$ as first function and $\frac{1}{\sqrt{2x-1}}$ as the 2nd function.

$$I = \left|(x^2+x)\int \frac{dx}{\sqrt{2x+1}}\right|_2^4 - \int_2^4 \left\{(2x+1)\int \frac{dx}{\sqrt{2x+1}}\right\} dx$$

Now $\int \frac{dx}{\sqrt{2x+1}} = \frac{(2x+1)^{-\frac{1}{2}+1}}{2.\frac{1}{2}} = \sqrt{2x+1}$

Therefore, $I = \left|(x^2+x)\sqrt{2x+1}\right|_2^4 - \int_2^4 (2x+1)\sqrt{2x+1}\,dx$

$= \left(60 - 6\sqrt{5}\right) - \int_2^4 (2x+1)^{3/2}\,dx$

$= \left(60 - 6\sqrt{5}\right) - \left|\frac{(2x+1)^{5/2}}{5}\right|_2^4$

$= 60 - 6\sqrt{5} - \frac{1}{5}\left(9^{5/2} - 5^{5/2}\right) = 60 - 6\sqrt{5} - \frac{243}{5} + 5\sqrt{5}$

$= \frac{57}{2} - \sqrt{5}.$

(iv) $I = \int_1^e \frac{e^x}{x}(1 + x\log x)dx = \int_1^e e^x\left(\frac{1}{x} + \log x\right)dx$

$= \int e^x[f'(x) + f(x)]\,dx$ where $f(x) = \log x$

$= e^x f(x) = e^x \log x$

Therefore,

$$I = \int_1^e \frac{e^x}{x}(1 + x\log x)dx = [e^x \log x]_1^e = e^x \log e - e\log 1 = e^x$$

2.7. Definite Integral as area under the curve.

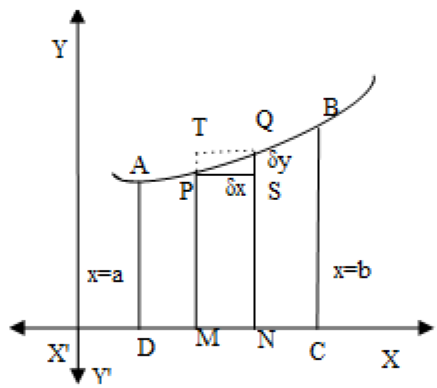
Let $f(x)$ be finite and continuous in $a \le x \le b$. Then area of the region bounded by x□axis, $y = f(x)$ and the ordinates at $x = a$ and $x = b$ is equal to $\int_a^b f(x)\ dx$.

Proof. Let AB be the curve $y = f(x)$ and $P(x, y)$ be any point on the curve such that $a \le x \le b$. Let DA and CB be the ordinates $x = a$ and $x = b$.

Take point $Q(x + \delta x, y + \delta y)$ near to the point $P(x, y)$. Draw PS and QT parallel to x-axis.

Clearly $PS = \delta x$ and $QS = \delta y$.

Let S represent the area bounded by the curve $y = f(x)$, x-axis and the ordinates AD $(x = a)$ and the variable ordinate PM.



Therefore, If δx is increment in x, then δS is increment in S.

It is clear from figure that δS is the area that lies between the rect. $PMNS$ and rect. $TQNM$.

Also area of rect. $PMNS = y.\delta x$ and area of rect. $TQNM = (y + \delta y)\, \delta x$

Therefore, $y\delta x < \delta S < (y + \delta y)\, \delta x$

Or $\qquad y < \frac{\delta S}{\delta x} < y + \delta y$

When $Q \to P$, $\delta x \to 0$, $\delta y \to 0$

And $\quad \lim_{\delta x \to 0} \frac{\delta S}{\delta x} \to \frac{dS}{dx}$, we get

$$\frac{dS}{dx} = y = f(x)$$

Therefore, $\int_a^b f(x)\, dx = \int_a^b \frac{dS}{dx}.dx = \int_a^b ds = |S|_a^b = (S)_{x=b} - (S)_{x=a}$

But it is clear from the figure, when $x = a, S = 0$, because then PM and AD coincide and then $x = b$, S = area $ABCD$ = reqd. area.

Therefore,

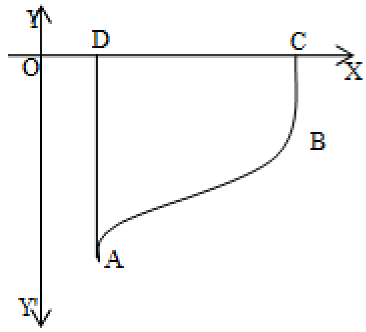
$$\int_a^b f(x)\, dx = \text{Area } ABCD.$$

Thus the area bounded by the curve $y = f(x)$, the x axis and the ordinates $x = a$ and $x = b$ is $\int_a^b f\,(x)\; dx$.

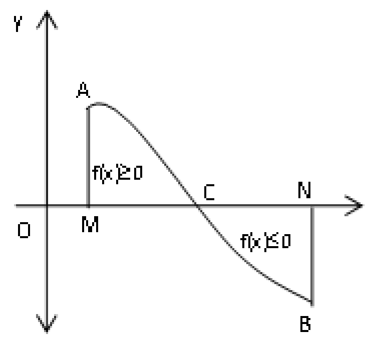
Remarks. In the figure given, we assumed that $f(x \geq 0)$ for all x in $a \leq x \leq b$. However, if

(i) $f(x) \leq 0$ for all x in $a \leq x \leq b$, then area bounded by x-axis, the curve $y = f(x)$ and the ordinate $x = a$ to $x = b$ is given by

$$= \int_a^b f\,(x)\; dx.$$



(ii) If $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $c \leq x \leq b$, then area bounded by $x = f(x)$, x-axis and the ordinates $x = a, x = b$, is

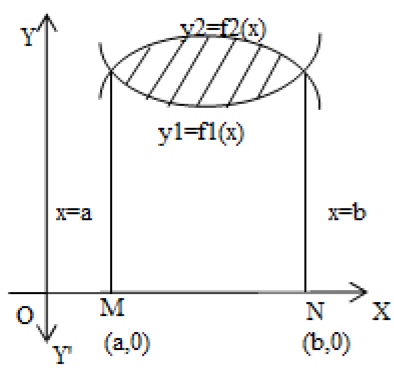


$$= \int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

$$= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

(iii) The area of the region bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and the ordinates $x = a$ and $x = b$ is given by

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$



where $f_2(x)$ is y_2 of upper curve and $f_1(x)$ is y_1 of lower curve i.e.,

$$\text{Required area} = \int_a^b [f_2(x) - f_1(x)]dx = \int_a^b (y_2 - y_1)\,dx.$$

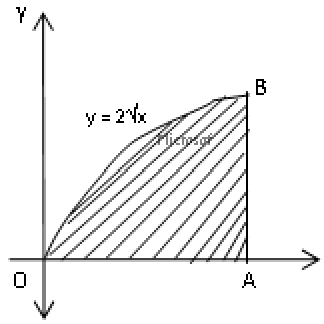
2.7.1. Example.

(a) Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

(b) Find the area under the curve $y = \sqrt{3x + 4}$ between $x = 0$ and $x = 4$.

Solution. (a) $y = 2\sqrt{x}$ implies $y^2 = 4x$

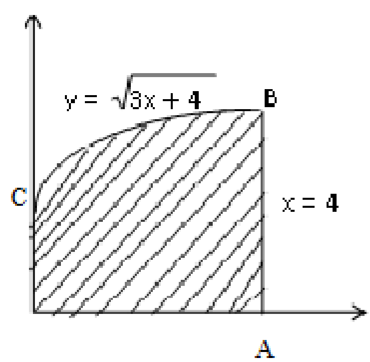
$y = 2\sqrt{x}$ is the upper part of the parabola $y^2 = 4x$. We have to find the area of the shaded region OAB.



Required area $= \int_0^1 y\,dx$

$$= \int_0^1 2\sqrt{x}\,dx = 2\left|\frac{x^{3/2}}{3/2}\right|_0^1 = \frac{4}{3} \text{ sq. units}$$

(b) $y = \sqrt{3x + 4}$, therefore, $y^2 = 3x + 4$. $y = \sqrt{3x + 4}$ is the upper part of the parabola $y^2 = 3x + 4$. We have to find the area of the shaded region.



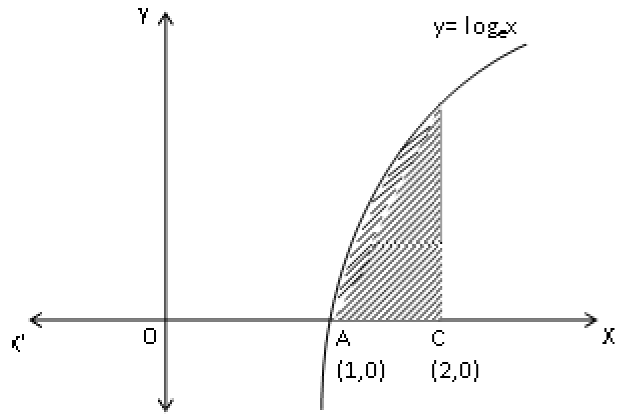
Required area OABC $= \int_0^4 y\,dx = \int_0^4 \sqrt{3x + 4}\,dx$

$$= \frac{2}{9}\left|(3x + 4)^{3/2}\right|_0^4 = \frac{112}{9} \text{ sq. units}$$

2.7.2. Example. Find the area bounded by $x = log_e\, x,\ y = 0$ and $x = 2$.

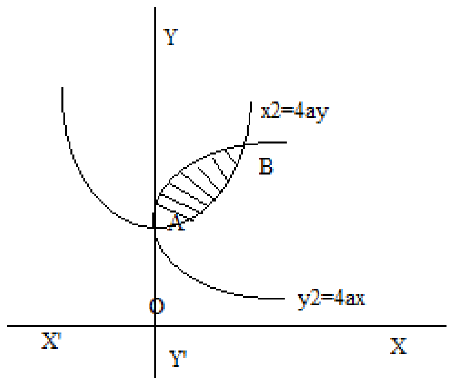
Solution. Required area ABC $= \int_1^2 y\,dx = \int_1^2 \log x\,dx$

$$= |x \log x - x|_1^2 = 2\log 2 - 1 = \log 4 - 1.$$



2.7.3. Example. Find the area included between two curves $y^2 = 4ax$ and $x^2 = 4\,ay$.

Solution. As shown in the figure, we have to find the area $OAPBO$.



Solving the given two equations simultaneously, we have

$$x^4 = 16a^2y^2 = 16a^2(4ax)$$

or $\quad x^4 = 64a^3x$ implies $x^4 - 64a^3x = 0,$

or $\quad x(x^3 - 64a^3) = 0$

or $\quad x = 0,\ x^3 = 64a^3$

Therefore, $x = 0$ at O and $x = 4a$ at B.

Now

Area $OAPBO$ = Area $OAPMO$ - Area $OBPMO$

$$= \int_0^{4a} y_1\,dx - \int_0^{4a} y_2\,dx = \int_0^{4a} 2a^{1/2}x^{1/2}\,dx - \int_0^{4a} \frac{x^2}{4a}\,dx$$

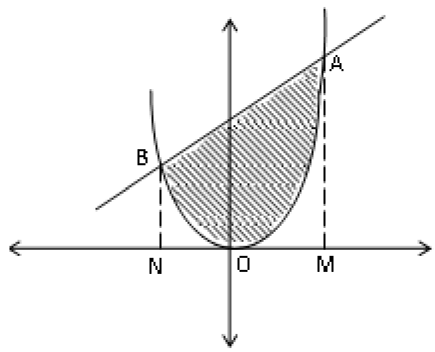
$$= 2a^{1/2}\int_0^{4a} x^{1/2}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2a^{1/2} \times \frac{2}{3}\left|x^{3/2}\right|_0^{4a} - \frac{1}{4a} \times \frac{1}{3}\left|x^3\right|_0^{4a}$$

$$= \frac{16}{3}a^2 \text{sq. units}$$

2.7.4. Example. Find the area cut-off from the parabola $4y = 3x^2$ by the straight line $2y = 3x + 12$.

Solution. Let the points of intersection of the parabola and the line be A and B as shown in the figure. Draw AM and BN perpendiculars to x-axis.



Now putting

$$y = \frac{3}{4}x^2 \text{ in } 2y = 3x + 12$$

We set $\frac{3}{2}x^2 = 3x + 12$

or $\quad 3x^2 - 6x + 24 = 0$

or $\quad x^2 - 2x - 8 = 0$

or $\quad x = 4,\ x = -2$

Then, $\quad y = 12,\ y = 3.$

The co-ordinates of the point A are $(4, 12)$ and co-ordinates of B are $(-2, 3)$.

Now, Required area AOB

= Area of trapezium $BNMA$ -[Area BNO + Area OMA]

But area of trapezium

$= \frac{1}{2}$(sumof ||sides)×Height

$= \frac{1}{2} \times (12 + 3) \times 6 = 15 \times 3 = 45.$

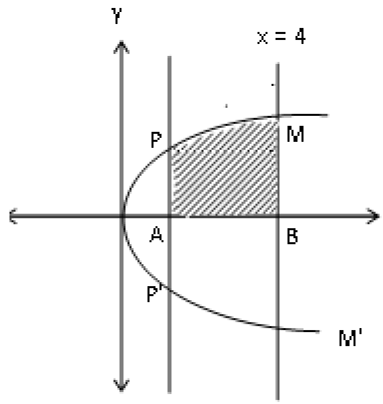
Area BNO + Area OMA $= \int_{-2}^{4} y\, dx$

$$= \frac{3}{4}\int_{-2}^{4} x^2\, dx = \frac{3}{4}\left|\frac{x^3}{3}\right|_{-2}^{4} = 18.$$

Hence required area $= 45 - 18 = 27$ sq. units.

2.7.5. Example. Find the area bounded by the parabola $y^2 = 2x$ and the ordinates $x = 1$ and $x = 4$.

Solution. The equation of the parabola is $y^2 = 2x$ which is of the form $y^2 = 4ax$. The parabola is symmetrical about x-axis and opens towards right.



In the first quadrant $y \geq 0$.

Required Area = $PMM'P'$

$$= 2 \text{ area } ABMP$$

$$= 2\int_1^4 y\,dx = 2\int_1^4 \sqrt{2}x^{1/2}\,dx$$

$$= 2\sqrt{2}\left|\frac{x^{3/2}}{3/2}\right|_1^4 = \frac{28\sqrt{2}}{3} \text{ sq. units.}$$

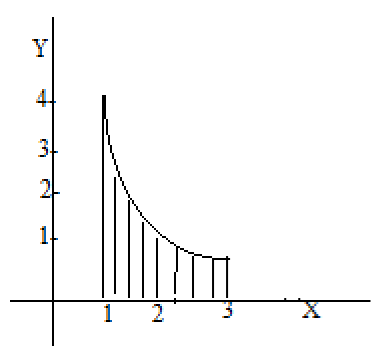
2.7.6. Example. Make a rough sketch of the graph of the function $y = \frac{4}{x^2}$ $(1 \leq x \leq 3)$, and find the area enclosed between the curve, the x-axis and the liens $x = 1$ and $x = 3$.

Solution. Given equation of the curve is

$$y = \frac{4}{x^2},\ (1 \leq x \leq 3)$$

Therefore, $y > 0$ i.e., the curve lies above the x-axis.

When $x = 1, 2, 3,$ then $y = 4, 1, 0.44$ respectively.

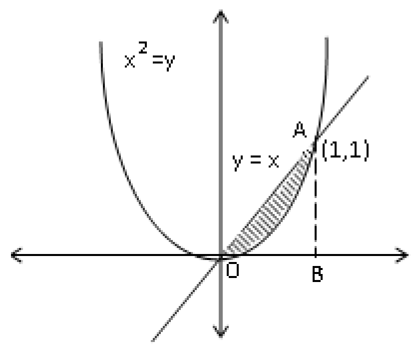


Required area $= \int_1^3 y\,dx$

$$= \int_1^3 \frac{4}{x^2}\,dx = 4\left|-\frac{1}{x}\right|_1^3 = \frac{8}{3} \text{ sq. units.}$$

2.7.7. Example. Find the area of the region $\{(x, y) : x^2 \leq y \leq x\}$.

Solution. Let us first sketch the region whose area is to be found out. The required area is the area included between the curves $x^2 = y$ and $y = x$.



Solving these two equations simultaneously, we have

$$x^2 = x \text{ implies } x^2 - x = 0$$

or $\quad x(x-1) = 0$

or $\quad x = 0, x = 1$

When $x = 0,\ y = 0$ and $x = 1,\ y = 1$.

Therefore, these two curves intersect each other at two points $O(0,0)$ and $A(1,1)$.

$$\text{Required area} = \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1 - \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{6} \text{ sq. units.}$$

2.7.8. Example. Find the area of the region $\{(x,y) : x^2 \le y \le |x|\}$.

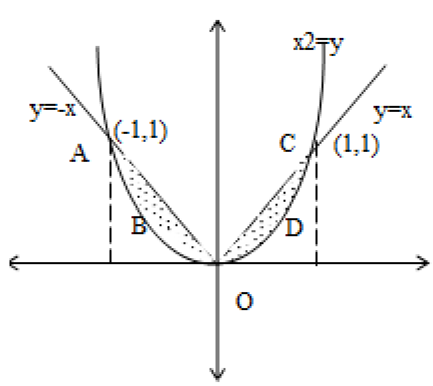
Solution. Let us first sketch the region whose area is to be found out.

The required area is the area included between the curves $x^2 = y$ and $y = |x|$.

The graph of $x^2 = y$ is a parabola with vertex (0, 0) and axis y-axis as shown in figure.

The graph of $y = |x|$ is the union of lines $y = x, x \ge 0$ and $y = -x, x \le 0$.

The required region is the shaded region.



Therefore, the required area = Area OAB+Area OCD

$$=2 \text{ Area OCD}$$

$$= 2\int_0^1 x\,dx - 2\int_0^1 x^2\,dx$$

$$= 2\left|\frac{x^2}{2}\right|_0^1 - 2\left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3} \text{ sq. units.}$$

2.7.9. Example. Using integration find the area of the triangular region whose sides have the Equation

$$y = 2x + 1 \quad \ldots(1)$$

$$y = 3x + 1 \quad \ldots(2)$$

and $x = 4$ ...(3)

Solution. Solving (1) and (3), we get $x = 4, y = 2 \times 4 + 1 = 9$.

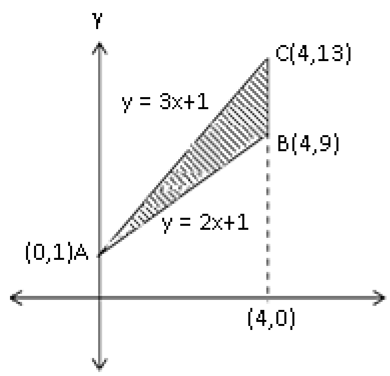
Therefore, $(4, 9)$ is the point of intersection of lines (1) and (3).

Solving (1) and (2), we get $x = 0,\ y = 1$.

Therefore, $(0, 1)$ is the point of intersection of lines (1) and (2).

Solving (2) and (3), we get $x = 4,\ y = 3 \times 4 + 1 = 13$.

Therefore, $(4, 13)$ is the point of intersection of lines (2) and (3)



Required area ABC $= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx$

$$= \left|\frac{3x^2}{2} + x\right|_0^4 - \left|\frac{2x^2}{2} + x\right|_0^4 = 8 \text{ sq. units.}$$

2.8. Learning Curve.

Learning curve is a technique with the help of which we can estimate the cost and time of production process of a product. With passage of time, the production process becomes increasingly mature and reaches a steady state. It so happens because with gain in experience with time, time taken to produce one unit of a product steadily decreases and in the last attains a stable value.

The general form of the learning curve is given by

$$y = f(x) = ax^{-b}$$

where y is the average time taken to produce one unit, and x is the number of units produced, a and b are the constants.

a is defined as the time taken for producing the first unit ($x = I$) and b is calculated by using the formula

$$b = \frac{\log(learning\ \ rate)}{\log 2}$$

If the learning curve is known, then total time (labour hours) required to produce units numbered from a to b is given by

$$L = \int_a^b f(x)\,dx = \int_a^b A.x^a\,dx \qquad \text{(another form of learning curve)}$$

2.8.1. Example. The first batch of 10 dolls is produced in 30 hours. Determine the time taken to produce next 10 dolls and again next 20 dolls, assuming a 60% learning rate. Estimate the time taken to produce first unit.

New Time taken to produce one batch

= Previous time taken to produce one batch × learning rate

No. of dolls	**Total time (hours)**	**Total increase in time**	**Average time**
0	0	-	-
10	30	30	3
20	$20\left(\frac{30\times 60}{100}\right) = 36$	6	1.8
40	$20\left(\frac{36\times 60}{100}\right) = 43.2$	7.2	1.08

Now, $\beta = -\frac{\log(0.6)}{\log 2} = -0.7369$

When $x = 10\,, y = 3$, then $3 = a.10^{-0.7369}$

Solving the equation, we get $A = 16.38$ hours.

2.8.2. Example. Because of learning experience, there is a reduction in labour requirement in a firm. After producing 36 units, the firm has the learning curve $f(x) = 1000\,x^{-1/2}$. Find the labour hours required to produce the next 28 units.

Solution L = $\int_{36}^{64} 1000x^{-0.5}\,dx$

$= 1000[2x^{1/2}]_{36}^{64} = 2000[8-6] = 4000$ hours.

2.8.3. Example. A firm's learning curve after producing 100 units is given by $f(x) = 2400\, x^{-0.5}$ which is the rate of labour hours required to produce the x^{th} unit. Find the hours needed to produce an additional 800 units.

Solution. Labour hours required

$$L = \int_{100}^{900} f(x)\, dx$$

$$= \int_{100}^{900} 2400x^{-0.5}\, dx = 2400\left[\frac{x^{1/2}}{1/2}\right]_{100}^{900} = 4800[30 - 10] = 96000 \text{ hours.}$$

2.9. Consumer and Producer Surplus.

Consumer surplus is the difference between the price that a consumer is willing to pay and the actual price he pays for a commodity. The degree of satisfaction derived from a commodity is a subjective matter.

If DD_1 is the market demand curve then demand x_0 corresponds to the price p_0. The consumer surplus is given by DD_1p_0.

$$DD_1p_0 = \text{Area } DD_1x_00 - \varphi_0D_1x_0O$$

$$= \int_0^{x_0} f(x)\, dx - p_0x_0$$

where $f(x)$ is the demand function.

It is assumed that the area is defined at $x = 0$ and that the satisfaction is measurable in terms of price for all consumers. In other words, we assume that utility function is same for all consumers and marginal utility of money is constant.

2.9.1. Example. Find the consumer surplus if the demand function is $p = 25 - 2x$ and the surplus function is $4p = 10 + x$.

Solution. First find the equilibrium price p_0 and equilibrium demand, x_0 by solving the above two equations simultaneously, we have

$$x_0 = 10 \text{ and } p_0 = 5$$

Now consumer surplus $= \int_0^{x_0} f(x)\, dx - p_0x_0$

$$= \int_0^{10} (25 - 2x)\, dx - 5 \times 10$$

$$= [25x - x^2]_0^{10} - 50 = 100.$$

2.10. Producer Surplus

Producer surplus is the difference in the prices a producer expects to get and the price which he actually gets for a commodity.

If SS_1 is the market supply curve and if x_0 is the supply at the market price p_0, the producer surplus is the area PS.

$$PS = \text{Area } SS_1P_0 = p_0x_0 - \int_0^x g(x)\,dx$$

where $g(x)$ is the supply function.

2.10.1. Example. Find the producer surplus for the supply function $p^2 - x = 9$ when $x_0 = 7$

Solution. We are given $p^2 - x = 9$ or $p_0^2 - x_0 = 9$

Also given $x_0 = 7$

Therefore, $p_0 = 7$

Thus,

$$\begin{aligned} PS &= p_0x_0 - \int_0^x g(x)\,dx \\ &= 4 \times 7 - \int_0^7 (x+9)^{1/2}\,dx \\ &= 28 - \left[\frac{2}{3}(x+9)^{3/2}\right]_0^7 = \frac{10}{3}. \end{aligned}$$

2.11. Leontief Input-Output Model.

Consider a model consisting of n production units and each unit produces only one kind of product. Each unit in the model uses the output of these n units as input. Also some part of output of each unit is used by other consumers, we shall call those parts as final demand of the unit. The sum of all the outputs of a particular unit is known as total output of that unit. Now, we have to determine the new total output of a unit if the final demand changes assuming that the resources of the model does not change. Here comes the role of Leontief input-output model. We illustrate the process for three production units.

1. Avaiblable data. Let P_1, P_2, P_3 be three production units and x_{ij} denote the part of output of the unit P_i used as input by the units P_j. Let b_i denotes the final demand of unit P_i and X_i denotes the total output of unit P_i. This data can be represented as:

Production Unit	Receiving unit			Final demand	Total output
	P_1	**P_2**	**P_3**		
P_1	x_{11}	x_{12}	x_{13}	b_1	X_1
P_2	x_{21}	x_{22}	x_{23}	b_2	X_2
P_3	x_{31}	x_{32}	x_{33}	b_3	X_3

2. Construction of Technology matrix. The ratio $\dfrac{x_{ij}}{X_j}$ is denoted by a_{ij} and is known as input-output coefficients or technical coefficients. For example,

$$a_{11} = \frac{x_{11}}{X_1},\ a_{12} = \frac{x_{12}}{X_2},\ a_{13} = \frac{x_{13}}{X_3}$$

Then the matrix A of all these input-output coefficients is called Technology matrix or matrix of technical coefficients. Thus the technical matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3. Simon-Hawkins Conditions. The conditions for the system to be viable are:

(i) The elements on the principal diagonal of Leontief matrix are all positive i.e., $1-a_{11}, 1-a_{22},...$ are all positive.

(ii) The determinant of Leontief matrix i.e., $|I-A|$ is positive.

If these two conditions are not satisfied, then there is no solution of the above system. These conditions are known as (Simon-Hawkins conditions for viability of system)

2.11.1. Example. For a two unit economy with production units X and Y, the inter unital demand and final demand are as follows :

Production Unit	Receiving unit		Final demand	Total output
	P_1	P_2		
P_1	30	40	50	120
P_2	20	10	30	60

(i) Find the technical coefficients.

(ii) Find the matrix of technical coefficients.

(iii) Find the Leontief matrix

(iv) Verify Simon-Hawkins conditions for viability of the system.

Solution. The given table is

Production Unit	**Receiving unit**		**Final demand**	**Total output**
	P_1	**P_2**		
P_1	$x_{11}=30$	$x_{21}=40$	$b_1=50$	$X_1=120$
P_2	$x_{12}=20$	$x_{22}=10$	$b_2=30$	$X_2=60$

(i) Here n = 2, technical co-efficients are $a_{11}\ a_{21}$, a_{12} , a_{22}

Thus
$$a_{11} = \frac{x_{11}}{X_1} = \frac{30}{120} = \frac{1}{4}$$
$$a_{21} = \frac{x_{21}}{X_1} = \frac{20}{120} = \frac{1}{6}$$
$$a_{12} = \frac{x_{12}}{X_2} = \frac{40}{60} = \frac{2}{3}$$
$$a_{22} = \frac{x_{22}}{X_2} = \frac{10}{60} = \frac{1}{6}$$

(ii) Matrix of technical coefficients $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

(iii) Leontief Matrix $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$

(iv) Elements on principal diagonal of Leontief matrix are $\frac{3}{4}$ and $\frac{5}{6}$ which are positive . Also, $|I - A|$

$$= \begin{vmatrix} \frac{3}{4} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{5}{6} \end{vmatrix} = \frac{37}{72} > 0$$

Hence Simon-Hawkins conditions are verified.

2.12. Check Your Progress.

1. Evaluate (i) $\int(4x^3 + 3x^2 - 2x + 5)dx$ (ii) $\int\left(\sqrt{x} - \frac{1}{2}x + \frac{2}{\sqrt{x}}\right)dx$

(iii) $\int\left(\frac{x^4+1}{x^2}\right)dx$ (iv) $\int\left(x - \frac{1}{x}\right)^3 dx$

(v) $\int\left(2^x + \frac{1}{2}e^{-x} + \frac{4}{x} - \frac{1}{\sqrt[3]{x}}\right)dx$ (vi) $\int\frac{x}{x-3}dx$

(vii) $\int\left(e^{a\log x} + e^{x\log a}\right)dx$ (viii) $\int\frac{1}{\sqrt{5x+3}-\sqrt{5x+2}}dx$

(ix) $\int\left(e^{3x} - 2e^x + \frac{1}{x}\right)dx$ (x) $\int\frac{(x^3+1)(x-2)}{x^2-x-2}dx$

(xi) $\int\frac{(a^x+b^x)^2}{a^x b^x}dx$ (xii) $\int\frac{1}{\sqrt{x+1}+\sqrt{x-1}}dx$

2. Evaluation (i) $\int\frac{3x+5}{(3x^2+10x+2)^{2/3}}dx$ (ii) $\int\frac{\sqrt{2+\log x}}{x}dx$

(iii) $\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$ (iv) $\int\frac{dx}{x^2-a^2}$

(v) $\int x(x^2+4)^5\,dx$ (vi) $\int\frac{8x^2}{(x^3+2)^3}dx$

(vii) $\int\frac{x^3}{(x^2+1)^3}dx$ (viii) $\int\frac{x+2}{\sqrt{x^2+4x+5}}dx$

(ix) $\int \frac{(x+1)(x+\log x)^3}{3x} dx$ (x) $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e} dx$

(xi) $\int \frac{1}{(1+e^x)(1+e^{-x})} dx$ (xii) $\int (x+1)2^{x^2+2x}\, dx$

3. Evaluate (i) $\int x^2 e^{3x}\, dx$ (ii) $\int x^n \log x\, dx$

(iii) $\int \frac{xe^x}{(x+1)^2} dx$ (iv) $\int \log x\, dx$

(v) $\int \sqrt{4x^2-9}\, dx$ (vi) $\int \frac{dx}{(x+1)\sqrt{x+2}}$

(vii) $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$ (viii) $\int \frac{dx}{(x^2-1)\sqrt{x^2+1}}$

(ix) $\int x \log(1+x)\, dx$ (x) $\int x^3 a^{x^2} dx$

4. Evaluate. (i) $\int \frac{x}{(x-1)(x-2)} dx$ (ii) $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

(iii) $\int \frac{3x+5}{x^4-x^3-x^2+1} dx$ (iv) $\int \frac{x^2+1}{(2x+1)(x+1)(x-1)} dx$

(v) $\int \frac{26x+6}{8-10x-3x^2} dx$ (vi) $\int \frac{2x^3-3x^2-9x+1}{2x^2-x-10} dx$

(vii) $\int \frac{dx}{(x+1)^2(x^2+1)} dx$ (viii) $\int \frac{dx}{(e^x-1)^2}$

(ix) $\int \frac{x^2+x+1}{(x-3)^3} dx$ (x) $\int \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)} dx$

5. Evaluate the following:

(i) $\int_2^4 (3x-2)^2\, dx$ (ii) $\int_6^{10} \frac{1}{x+2} dx$ (iii) $\int_3^{11} \sqrt{2x+3}\, dx$

(iv) $\int_0^2 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{2-x}} dx$ (v) $\int_0^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$ (vi) $\int_0^1 \frac{3x^3-4x^2+1}{\sqrt{x}} dx$

6. Evaluate the following :

(i) $\int_0^1 \frac{x^5}{1+x^6} dx$ (ii) $\int_1^2 x\sqrt{3x-2}\, dx$ (iii) $\int_2^4 \frac{6x^2-1}{\sqrt{2x^3-x-2}} dx$

(iv) $\int_1^2 \frac{(\log x)^2}{x} dx$

7. Evaluate the following :

(i) $\int_0^1 xe^x\, dx$ (ii) $\int_0^1 x log\left(1+\frac{x}{2}\right) dx$ (iii) $\int_0^1 x^2 e^x\, dx$

(iv) $\int_a^b \frac{\log x}{x^2} dx$

8. Find the area of the region included between the parabola $y = \frac{3}{4}x^2$ and the line $3x - 2y + 12 = 0$.

9. Find the area bounded by the curve $y = x^2$ and the line $y = x$.

10. Make a rough sketch of the graph of the function $y = 9 - x^2,\ 0 \le x \le 3$ and determine the area enclosed between the curve and the axis.
11. Using integration, find the area of the region bounded by the triangle whose vertices are $(1,0), (2,2)$ and $(3,1)$.
12. Find the area of the region bounded by $y = -1,\ y = 2,\ x = y^2,\ x = 0$.
13. Find the area between the parabola $y^2 = x$ and the line $x = 4$.
14. Find the area bounded by the curve $y = x^2 - 4$ and the lines $y = 0$ and $y = 5$.
15. Find the area of the region enclosed between the curve $y = x^2 + 1$ and the line $y = 2x + 1$.
16. Find the area bounded by the curve $x = at^2,\ y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.
17. Find the area of the region enclosed by the parabola $y^2 = 4ax$ and chord $y = mx$.

Answers.

1. (i) $x^4 + x^3 - x^2 + 5x + C$ (ii) $\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4\sqrt{x} + C$

(iii) $\frac{x^3}{3} - \frac{1}{x} + C$ (iv) $\frac{x^4}{4} - \frac{3}{2}x^2 + 3\log x + \frac{1}{2x^2} + C$

(v) $\frac{2^x}{log\, 2} - \frac{1}{2}e^{-x} + 4\log x - \frac{3}{2}x^{2/3} + C$ (vi) $x + \log|x - 3| + C$

(vii) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log\, a} + C$ (viii) $\frac{2}{15}\left[(5x+3)^{\frac{3}{2}} - (5x+2)^{\frac{3}{2}}\right] + C$

(ix) $\frac{e^{3x}}{3} - 2e^x + \log|x| + C$ (x) $\frac{x^3}{3} - \frac{x^2}{2} + x + C$

(xi) $\frac{\left(\frac{a}{b}\right)^x}{\log\frac{a}{b}} + 2x + \frac{\left(\frac{b}{a}\right)^x}{\log\frac{b}{a}} + C$ (xii) $\frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C$

2. (i) $\frac{3}{2}(3x^2 + 10x + 2)^{\frac{1}{3}} + C$ (ii) $\frac{2}{3}(2 + \log x)^{\frac{3}{2}} + C$ (iii) $\log|e^x + e^{-x}| + C$

(iv) $\frac{1}{2a}log\frac{|x-a|}{|x+a|} + C$ (v) $\frac{1}{12}(x^2 + 4)^6$ (vi) $\frac{4}{3(x^2+2)^2}$

(vii) $-\frac{1}{4}\frac{2x^2+1}{(x^2+1)^2}$ (viii) $\frac{1}{15}(1 + x^6)^{5/2} + C$ (ix) $\frac{1}{12}(x + \log x)^4 + C$

(x) $\frac{1}{e}\log|e^x + x^e| + C$ (xi) $-\frac{1}{1+e^x} + C$ (xii) $\frac{2^{x^2+2x}}{2\log 2} + C$

3. (i) $\frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27}e^{3x}$ (ii) $\log x\ \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}$ (iii) $\frac{1}{x+1}e^x$

(iv) $x(\log x)^2 - 2xlog\ x + 2x$ (v) $\frac{\sqrt{4x^2-9}}{2} - \frac{9}{4}\log|2x + \sqrt{4x^2 - 9}| + C$

(vi) $\log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right| + C$ (vii) $1 - \log\left|\frac{1}{x+1} - \frac{1}{2} + \frac{\sqrt{x^2+x+1}}{x+1}\right| + C$

(viii) $-\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2x}+\sqrt{x^2+1}}{\sqrt{2x}-\sqrt{x^2+1}}\right| + C$ (ix) $\frac{1}{2}(x^2 - 1)\log(1 + x) - \frac{1}{4}x^2 + \frac{1}{2}x + C$

(x) $\frac{x^2 a^{x^2}}{2\log a} - \frac{a^{x^2}}{2(\log a)^2} + C$

4. (i) $-\log|x-1| + 2\log|x-2| + C$ (ii) $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$

(iii) $\frac{1}{2}log\ \left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$ (iv) $-\frac{5}{6}\log|2x+1| + \frac{1}{3}\log|x-1| + \log|x+1| + C$

(v) $-\frac{5}{3}\log|3x-2| - 7\log|x+4| + C$ (vi) $\frac{x^2}{2} - x + \log\left|\frac{x+2}{2x-5}\right| + C$

(vii) $\frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log|x^2+1| + C$

(viii) $\log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + C$ (ix) $\log|x-3| - \frac{7}{x-3} - \frac{13}{2(x-3)^2} + C$

(x) $a^3 + ab + clog\ |x-a| + \frac{ab^2+b^2+c}{(b-a)(b-c)}\log|x-b| + \frac{c(ac+b+1)}{(c-a)(c-b)}\log|x-c| + k$

5. (i) 104 (ii) $\log\frac{3}{2}$ (iii) $\frac{98}{3}$ (iv) 1 (v) $\frac{1}{\sqrt{3}}$ (vi) $-\frac{52}{15}$

6. (i) $\frac{1}{6}\log 2$ (ii) $\frac{326}{135}$ (iii) $2(\sqrt{122} - \sqrt{12})$ (iv) $\frac{1}{3}(\log 2)^3$

7. (i) 1 (ii) $\frac{3}{4} + \frac{2}{3}\log\frac{2}{3}$ (iii) e^{-2} (iv) $\frac{\log a+1}{\log a} - \frac{\log b+1}{\log b}$

8. 27 9. $\frac{1}{6}$ 10. 18 11. $\frac{3}{2}$

12. $\frac{15}{4}$ 13. $\frac{32}{3}$ 14.. $\frac{76}{3}$ 15. $\frac{4}{3}$

16. $\frac{56a^2}{3}$ 17. $\frac{8a^2}{3m^2}$

2.13. Summary. In this chapter, we derived methods to find the integration of various functions by using various methods. Also, Leontiff input-output model is discussed.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.

3

Matrices

Structure

3.1. Introduction. In 1857, Arthur Cayley, formulated the general theory of matrices. In the study of mathematics matrices have the most important place and found useful in many branches of science, engineering, economics and commerce. This chapter contains many important results related to matrices, their addition subtraction, multiplication, various types and their realtions.

3.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Matrices.

(ii) Various types of matrices.

(iii) Algebra of matrices.

3.1.2. Keywords. Matrix, Comparable Matrices, Symmetric Matrix, Non-Symmetric Matrix.

3.2. Matrices.

3.2.1. Matrix. A rectangular representation of numbers (data) or functions is known as a matrix. A matrix is always represented by capital letters *A*, *B*, *C*, ... etc.

For example $A=\begin{bmatrix}1 & 1 & 6\\ 2 & 0 & 3\end{bmatrix}$ is a matrix.

In a matrix, horizontal lines are called **rows** and vertical lines are called **columns**. For example**,**

$$A=\underbrace{\begin{bmatrix}1 & 1 & 6\\ 2 & 0 & 3\end{bmatrix}}_{\text{Columns}}\left.\begin{matrix}\to\\ \to\end{matrix}\right\}\text{Rows}$$

A matrix may have any number of rows and columns.

3.2.2. Order of matrix. By stating that A is a matrix of order m x n, we mean that the matrix A is having m rows and n columns.

Generally, a matrix is represented as $A=\left[a_{ij}\right]_{mxn}$, which is a matrix of order m x n, having m rows and n columns. a_{ij} 's are known as elements of the matrix A. In particular a_{ij} is the jth entry in the ith row.

3.2.3. Types of Matrices.

Here we are discussing some useful types of matrices:

3.2.4. Square matrix. A matrix of order $m \times n$ is called a square matrix if $m = n$, that is, if the number of rows is equal to number of columns

For example**.** $A=\begin{bmatrix}1 & 1 & 8\\ 0 & 0 & 0\\ 2 & 1 & 5\end{bmatrix}$ is a square matrix of order 3.

The elements $a_{11}, a_{22}, \ldots$ are called the diagonal elements of matrix A . Thus, those a_{ij} for which $i = j$ are called diagonal elements. Rest of the elements are called the non-diagonal elements of square matrix A, that is, those a_{ij} for which $i \neq j$.

3.2.5. Row matrix. A matrix having single row and any number of columns is called a row matrix. For example, $A=[1\ \ 0\ \ 4\ \ 7\ \ 5\ \ 6]$ is a row matrix of order 1×6.

3.2.6. Column matrix. A matrix having single column and any number of rows is called column matrix.

For example, $A=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$ is a column matrix of order 3×1.

3.2.7. Zero or null matrix. A matrix is said to be a zero or null matrix if all its elements are zero. Usually it is denoted by O. For example, $A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is a zero matrix.

3.2.8. Diagonal matrix. A square matrix in which all non-diagonal elements are zero is called a diagonal matrix. So, $A=\left[a_{ij}\right]_{m\times n}$ is a diagonal matrix if a_{ij}=0 for $i \neq j$.

For example, $A=\begin{bmatrix}3 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 7\end{bmatrix}$ is a diagonal matrix of order 3 × 3

It should be noted that the diagonal elements in a diagonal matrix may or may not be zero. Further, a diagonal matrix of order n can be denoted as $A = diag.[a_{11} \quad a_{22} \quad . \quad . \quad . \quad a_{nn}]$.

3.2.9. Scalar matrix. A diagonal matrix in which all its diagonal elements are equal is called a scalar matrix**.** For example**,** $A = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$ is a scalar matrix of order 3 × 3. Thus, a diagonal matrix of order n can be denoted as $A = diag.[a \quad a \quad . \quad . \quad . \quad a]$.

Note. All square zero matrices are always diagonal as well as scalar matrix**.**

3.2.10. Unit matrix or Identity matrix. A scalar matrix with all entries '1' is called an identity matrix. Usually a unit matrix is denotes by I_n where n represents order of matrix.

For example, $I_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $I_3 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

3.2.11. Triangular Matrices.

(i) Upper triangular matrix. A matrix in which all elements below the principal diagonal are zero is called an upper triangular matrix. For example**,** $A=\begin{bmatrix}1 & 0 & 4\\ 0 & 2 & 7\\ 0 & 0 & 4\end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

(ii) Lower triangular matrix. A matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix. For example**,** $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 6 & 0\\ 4 & 2 & 7\end{bmatrix}$ is a lower triangular matrix of order 3 × 3.

Remark. Diagonal matrices are upper triangular as well as lower triangular matrices.

3.2.12. Comparable matrices. Two matrices A and B are comparable if their orders are same, that is, if A be a matrix of order $m \times n$ and B be a matrix of order $p \times q$ then A and B are comparable if $m = p$ and $n = q$

3.2.13. Equal matrices. Two matrices are equal if they are of same order and having same elements in the corresponding positions. For example, if $\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & 4\\ 0 & 6\end{bmatrix}$ then $a = 1$, $b = 4$, $c = 0$, $d = 6$.

3.2.14. Example. Construct a 2 × 3 matrix $A = [a_{ij}]$ whose element a_{ij} are given by

$$a_{ij} = \frac{(i+2j)^2}{3}$$

Solution. Let $A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}$.

Given that $a_{ij} = \frac{(i+2j)^2}{3}$

Then, $a_{11} = \frac{[(1)+2(1)]^2}{3} = 3$, $a_{12} = \frac{[(1)+2(2)]^2}{3} = \frac{25}{3}$, $a_{13} = \frac{[(1)+2(3)]^2}{3} = \frac{49}{3}$

$a_{21} = \frac{[(2)+2(1)]^2}{3} = \frac{16}{3}$, $a_{22} = \frac{[(2)+2(2)]^2}{3} = 12$, $a_{23} = \frac{[(2)+2(3)]^2}{3} = \frac{64}{3}$

Therefore the required matrix is $A = \begin{bmatrix} 3 & \frac{25}{3} & \frac{49}{3} \\ \frac{16}{3} & 12 & \frac{64}{3} \end{bmatrix}$.

3.2.15. Exercise. If $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ then find *x*, *y*, *z* and *w*.

Solution. Given that $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Comparing the corresponding elements, we obtain

$$x - y = -1$$

$$2x + z = 5$$

$$2x - y = 0$$

$$3z + w = 13$$

Subtracting first and third equation we can obtain

$$x = 1$$

Using $x = 1$ in first equation, we get

$$y = 2$$

Then from second equation, we get

$$2 + z = 5 \quad \Rightarrow \quad z = 3$$

Then from fourth equation, we get

$$9 + w = 13 \quad \Rightarrow \quad w = 4$$

Hence $x = 1$, $y = 2$, $z = 3$, $w = 4$

3.2.16. Exercise.

1. Construct a 3×3 matrix $A = [a_{ij}]$ whose element a_{ij} is given by $\frac{|-3+i+j|}{2}$.
2. Give an example of a matrix which is diagonal but not scalar.
3. For what values of *a* and *b* the following matrices are equal

$$A = \begin{bmatrix} 2a+1 & 3b \\ 5 & b^2 - 5b \end{bmatrix} \quad B = \begin{bmatrix} a+3 & b^2+2 \\ 5 & -6 \end{bmatrix}$$

4. Find the values of a and b if

(i) $\begin{bmatrix} a & 5 \\ 6 & b \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} a+b & 8 \\ 6 & ab \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 6 & 8 \end{bmatrix}$

5. For what values of a and b are the matrices $A = \begin{bmatrix} a+3 & b^2 \\ 0 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2a+1 & 2b \\ 0 & b^2-5b \end{bmatrix}$ are equal?

Answer.

2. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. $a = 2$, $b = 2$

4. (i) $a = 7$, $b = 4$ (ii) $a = 2$, $b = 4$, or $a = 4$, $b = 2$,

5. $a = 2$, $b = 2$

3.3. Sum, Difference and Scalar Multiplication of Matrices.

3.3.1. Addition of Matrices.

If A and B are two matrices of same order, then their sum $A+B$ is obtained by adding the corresponding elements of A and B. Clearly order of $A + B$ is similar to that of A and B.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 3 \end{bmatrix}_{2\times 3}$. Then, $A+B = \begin{bmatrix} 4 & 2 & 4 \\ 4 & 7 & 6 \end{bmatrix}_{2\times 3}$.

Note. Addition of two or more matrices is defined only when they are comparable otherwise sum of two matrices does not exist.

3.3.2. Properties of matrix addition

If A and B are two matrices of same order then following properties holds:

(i) Matrix addition is commutative **that is,** , $A + B = B + A$

(ii) Matrix addition is associative **that is,** , $(A + B) + C = A + (B + C)$

3.3.3. Difference of matrices.

For two matrices A and B of the same order, then their difference A - B is obtained by subtracting the elements of B from the corresponding elements of A. Clearly order of A - B is similar to that of A and B.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 5 & 3 \end{bmatrix}_{2\times 3}$. Then, $A-B = \begin{bmatrix} -2 & 0 & -4 \\ 0 & -3 & 0 \end{bmatrix}_{2\times 3}$.

3.3.4. Scalar Multiplication and its Properties.

If A is any matrix of order $m \times n$ and k is any scalar then kA is obtained by multiplying every element of A with k and known as scalar multiple of A by k.

For example, let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}_{2\times 3}$, $k = 3$. Then, $kA = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 9 \end{bmatrix}_{2\times 3}$.

3.3.5. Properties of Scalar Multiplication.

If A and B are two matrices of same order and k is any scalar then following properties holds**.**

(i) $k(A+B) = kA + kB$ **(ii)** $kl(A) = k(lA) = l(kA)$

(iii) $(k+l)A = kA + lA$

3.3.6. Example. If A = $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 4 \end{bmatrix}$ **and B =** $\begin{bmatrix} 0 & 1 & 4 \\ -2 & 3 & -2 \end{bmatrix}$ **, then find $A + B$ and $A - B$.**

Solution. Since the order of A and B are same, therefore addition and subtraction are possible. Then by definition

$$A + B = \begin{bmatrix} 1+0 & 5+1 & 6+4 \\ 2-2 & 4+3 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 10 \\ 0 & 7 & 2 \end{bmatrix}$$

and $$A - B = \begin{bmatrix} 1-0 & 5-1 & 6-4 \\ 2+2 & 4-3 & 4+2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 6 \end{bmatrix}.$$

3.3.7. Example. If A = $\begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \end{bmatrix}$ **and B =** $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{-1} \end{bmatrix}$**, then find a matrix C such that** $A+B+C$ **is a zero matrix.**

Solution. Given that A + B + C = 0

$$\Rightarrow \quad C = -A - B = -(A + B).$$

Here, $$A + B = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

Therefore, $$C = (-1)\,(B + A) = \begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}.$$

3.3.8. Example. Find x and y, if $\mathbf{2}\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$**.**

Solution. Given that $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

which implies $$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

which implies $$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Then, we have

$2x + 3 = 7$ and $2y - 4 = 14$

which implies $2x = 4$ and $2y = 18$

or $x = 2$ and $y = 9$

So $x = 2, y = 5$.

3.3.9. Exercise.

1. If $A=\begin{bmatrix}2 & 3\\ 4 & 6\end{bmatrix}$ then find $3A$ and $-2A$.

2. Compute $A+B$ if defined for the following

   (i) $A=\begin{bmatrix}-1 & 4 & 7\\ 8 & 5 & 1\\ 2 & 8 & 5\end{bmatrix}$, $B=\begin{bmatrix}2 & 3 & 1\\ 8 & 0 & 5\\ 3 & 2 & 4\end{bmatrix}$

   (ii) $A=\begin{bmatrix}-1 & 2 & 3\\ 4 & -5 & 6\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & -3\\ 0 & 5 & 0\end{bmatrix}$

3. If $\begin{bmatrix}\mathbf{1} & \mathbf{-5}\\ \mathbf{6} & \mathbf{7}\end{bmatrix}+\boldsymbol{X}=\begin{bmatrix}\mathbf{2} & \mathbf{-7}\\ \mathbf{1} & \mathbf{8}\end{bmatrix}$, then find X.

4. If $A=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-3}\\ \mathbf{5} & \mathbf{0} & \mathbf{2}\\ \mathbf{1} & \mathbf{-1} & \mathbf{1}\end{bmatrix}$, $B=\begin{bmatrix}\mathbf{3} & \mathbf{-1} & \mathbf{2}\\ \mathbf{4} & \mathbf{2} & \mathbf{5}\\ \mathbf{2} & \mathbf{0} & \mathbf{3}\end{bmatrix}$. Find a matrix C such that $A+2C=B$.

5. If $2X+3Y=\begin{bmatrix}\mathbf{2} & \mathbf{3}\\ \mathbf{4} & \mathbf{0}\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}\mathbf{-2} & \mathbf{2}\\ \mathbf{1} & \mathbf{-5}\end{bmatrix}$. Find X and Y.

6. Find X and Y if $2X+Y=\begin{bmatrix}\mathbf{4} & \mathbf{4} & \mathbf{7}\\ \mathbf{7} & \mathbf{3} & \mathbf{4}\end{bmatrix}$ **and** $\boldsymbol{X}-\mathbf{2}\boldsymbol{Y}=\begin{bmatrix}\mathbf{-3} & \mathbf{2} & \mathbf{1}\\ \mathbf{1} & \mathbf{-1} & \mathbf{2}\end{bmatrix}$.

7. If $A=\begin{bmatrix}2 & 1 & 3\\ 4 & 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}0 & 5 & 1\\ 2 & 3 & 1\end{bmatrix}$ then find C if $A+B+C$ is zero matrix.

Answer.

1. (i) $\begin{bmatrix}6 & 9\\ 12 & 18\end{bmatrix}$ (ii) $\begin{bmatrix}-4 & -6\\ -8 & -12\end{bmatrix}$

2. (i) $\begin{bmatrix}1 & 7 & 8\\ 16 & 5 & 6\\ 5 & 10 & 9\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 3 & 0\\ 4 & 0 & 6\end{bmatrix}$

3. $X=\begin{bmatrix}1 & -2\\ -5 & 1\end{bmatrix}$

4. $C=\begin{bmatrix}1 & -3/2 & 5/2\\ -1/2 & 1 & 3/2\\ 1/2 & 1/2 & 1\end{bmatrix}$.

5. $X=\begin{bmatrix}-2 & 0\\ -1 & -3\end{bmatrix}$, $Y=\begin{bmatrix}2 & 1\\ 2 & 2\end{bmatrix}$.

6. $X=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$ and $Y=\begin{bmatrix}2 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$

7. $\begin{bmatrix}-2 & -6 & -4\\ -6 & -5 & -6\end{bmatrix}$

3.4. Multiplication of Matrices.

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$ and $b = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix}_{n\times p}$ be two matrices of orders $m \times n$ and $n \times p$ respectively.

Then their product AB is a matrix C of order $m \times p$ and can be obtained as

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots + a_{in}b_{nj} \qquad \text{for} \qquad 1 \le i \le m,\ 1 \le j \le p$$

where $C = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}_{m\times p}$. For an example, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$.

Then AB can be obtained as follows:

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ \to & \to \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} \downarrow & b_{12} \downarrow & b_{13} \\ b_{21} \downarrow & b_{22} \downarrow & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}.$$

Remark. If the number of columns of A are not equal to number of rows of B then the product AB is not defined. For the above example, the product BA is not possible as number of columns in B (3) is not equal to number of rows in A (2).

3.4.1. Properties of matrix multiplication.

1. Matrix multiplication is not commutative in general, that is, AB may or may not equal to BA.
2. Matrix multiplication is associative, that is, if A, B and C are matrices of order $m \times n$, $n \times p$ *and* $p \times q$ *respectively, then* $(AB)C = A(BC)$.
3. Matrix multiplication is distributive over addition, that is, if A, B and C are matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively, then $A(B+C) = AB+AC$.
4. If A and B are n-rowed matrices then

   i) $(A+B)^2 = A^2 + B^2 + AB + BA$

   ii) $(A-B)^2 = A^2 + B^2 - AB - BA$

   iii) $(A+B)(A-B) = A^2 - AB + BA - B^2$

3.4.2. Matrices and Polynomials.

If $f(x)$ is a polynomial of degree n

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n.$$

and A is a square matrix of order m, then $f(A)$ is defined as:

$$f(A) = a_0 A^n + a_1 A^{n-1} + \ldots\ldots + a_{n-1} A + a_n I_m$$

3.4.3. Example. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$, then Compute AB.

Solution. By definition of product of two matrices

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 2+6+12 & 5+12+21 & 3+8+15 \\ 8+15+24 & 20+30+42 & 12+20+30 \end{bmatrix} = \begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}.$$

3.4.4. Example. If $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$, then find x and y.

Solution. Given that $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. Using the definition of product of two matrices

$$\begin{bmatrix} 3x-4y \\ x+2y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

On comparing the corresponding elements, we get

$$3x-4y = 3$$

$$x+2y = 11$$

Solving these two, we get $x = 5$ and $y = 3$.

3.4.5. Example. If $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \mathrm{A} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$ then find A.

Solution. Since the product matrix is a 3×3 matrix and the first matrix in product is of order 3×2. Therefore, A must be a 2×3 matrix. So let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then, the given matrix equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

which implies $$\begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Comparing corresponding elements, we get

$$2a-d = -1,\ a = 1,\ -3a + 4d = 9,$$

$$2b-e = -8,\ b = -2,\ -3b + 4e = 22,$$

$$2c-f = 10,\ c = -5,\ -3c + 4f = 15.$$

Solving these equations, we get

$$a = 1, d = 3;\ b = -2, e = 4;\ c = -5, f = 0.$$

Hence, $$A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}.$$

3.4.6. Example. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$ and hence evaluate A^4.

Solution. Here, $A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3.3+1.(-1) & 3.1+1.2 \\ -1.3+2.(-1) & -1.1+2.2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$.

Thus, $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $A^2 - 5A + 7I = O$

Now, $A^2 = 5A - 7I \quad \Rightarrow \quad A^4 = A^2.A^2 = (5A-7I).(5A-7I) = 25A^2 - 35A - 35A + 49I$

$$= 25(5A-7I) - 70A + 49I = 125A - 175I - 70A + 49I$$

$$= 55A - 126I$$

$$= 55\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 126\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 165 & 55 \\ -55 & 110 \end{bmatrix} - \begin{bmatrix} 126 & 0 \\ 0 & 126 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}.$$

3.4.7. Example. A trust fund has Rs. 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs. 30,000 among the two types of bonds if the trust fund must obtain an annual total interest of Rs. 1800.

Solution. Total fund is Rs. 30,000. Let investment made in first bond is Rs. x, then investment in the second bond Rs. $30,000 - x$. As per given data

Annual interest on first bond = **5% =** $\frac{5}{100}$ per rupee.

Annual interest on second bond = 7% = $\frac{7}{100}$ per rupee.

Let A be the investment matrix, then A = $[x \quad 30,000 - x]$, and B be the annual interest per rupee matrix, thus B = $\begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}$.

Therefore, total annual interest can be obtained by AB = $[x \quad 30,000 - x]\begin{bmatrix} 5/100 \\ 7/100 \end{bmatrix}$

$$= \left[\frac{5x}{100} + \frac{7(30000-x)}{100}\right]$$

$$= \left[\frac{5x + 210000 - 7x}{100}\right] = \left[\frac{210000 - 2x}{100}\right]$$

Therefore, total annual interest is $\frac{210000-2x}{100}$.

Now for a total annual interest of Rs. 1800 , we must have

$$\frac{210000-2x}{100}=1800$$

which implies, $210000-2x=180000$

which implies, $2x=30000$

which implies, $x=15000$

Hence the investments in both bonds are Rs. 15000.

3.4.8. Exercise.

1. Find the order of $A\times B$ if the order of A and B are

   (i) 2×3 and 3×2 (ii) 5×4 and 4×3.

2. If $A=\begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}a & 0\\ b & -1\end{bmatrix}$ and $(A+B)^2=A^2+B^2$. Find a and b.

3. Give examples of matrices

   (i) A and B such that $AB=O$, but $A\neq 0$, $B\neq 0$

   (ii) A, B, C, such that $AB=AC$, but $B\neq C$; $A\neq 0$

4. If $A=[1\ \ 2\ \ 3]$, $B=\begin{bmatrix}2 & 1\\ 0 & 3\\ 5 & 4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2\\ 3 & 2\\ 1 & 0\end{bmatrix}$. Verify that $A(B+C)=AB+AC$.

5. Solve the matrix equation $[x\quad 1]\begin{bmatrix}1 & 0\\ -2 & -3\end{bmatrix}\begin{bmatrix}x\\ 5\end{bmatrix}=0$.

6. If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}$, then find k so that $A^2=8A+kI_2$.

7. A bookseller furnished a school with 10 dozen books for class X, 8 dozen books for class XI and 5 dozen books for class XII. If their prices are Rs. 83, Rs. 34.50 and Rs. 45 respectively per book, find the total amount of the bill furnished by the bookseller.

8. There are three families. Family A consists of 2 men, 3 women and 1 child. Family B has 2 men, 1 woman and 3 children. Family C has 4 men, 2 women and 6 children. Daily income of a man and woman are Rs. 20 and Rs. 15.50 respectively and children have no income. Using matrix multiplication, calculate the daily income of each family.

Answers.

1. (i) 2×2 (ii) 5×3. **5.** $x=5$ or $x=-3$. **6.** $k=-7$. **7.** Rs. 15972

8. Rs. 86.50, Rs. 55.50, Rs. 111

3.5. Transpose of a Matrix.

Let A be any $m \times n$ matrix, then any $n \times m$ matrix obtained from A by changing its rows into columns or columns into rows is called the transpose of A and is denoted by A^T or A'.

For example, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3}$ then $A' = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}_{3\times 2}$.

3.5.1. Properties of Transpose.

1. $(A+B)' = A' + B'$ (A and B being same order)
2. $(A')' = A$
3. $(k\,A)' = k\,A'$ (where k is any scalar)
4. $(AB)' = B'A'$ (A and B being conformable for product).

3.6. Symmetric and Skew Symmetric Matrices.

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ is called a **symmetric matrix** if $A' = A$ or $a_{ij} = a_{ji}$ for all i and j.

For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 8 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 8 \end{bmatrix}$, then $A' = A$ and hence A is a symmetric matrix.

A square matrix $A = \left[a_{ij}\right]_{n\times n}$ is called a **skew- symmetric matrix** if $A' = -A$ or $a_{ij} = -a_{ji}$ for all i and j.

For example, let $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ then $A' = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix} = -A$. Hence A is skew-symmetric matrix.

3.6.1. Theorem. The diagonal elements of a skew symmetric matrix are all zero.

Proof. Let $A = \left[a_{ij}\right]$ be a skew symmetric matrix. Then by definition of skew symmetric matrix, $a_{ij} = -a_{ji}$ for all i, j. However, for diagonal elements $i = j$,

$$\Rightarrow \qquad a_{ii} = -a_{ii} \text{ for all values of } i$$

$$\Rightarrow \qquad 2a_{ii} = 0 \text{ for all } i\,.$$

$$\Rightarrow \qquad a_{ii} = 0 \text{ for all } i$$

Hence diagonal elements of a skew symmetric matrix are all zero.

3.6.2. Theorem. If A is any square matrix , then

1. $\frac{1}{2}(A + A')$ is symmetric matrix.
2. $\frac{1}{2}(A - A')$ is skew symmetric.
3. AA' and $A'A$ are symmetric matrix.

Proof.

1. Let $K = \frac{1}{2}(A + A')$, then as $(A+B)' = A' + B'$ and $(A')' = A$, therefore

$$K' = \frac{1}{2}(A + A')' = \frac{1}{2}\left[A' + (A')'\right] = \frac{1}{2}(A' + A) = \frac{1}{2}(A + A') = K$$

So, $K = \frac{1}{2}(A + A')$ is a symmetric matrix.

2. Let $K = \frac{1}{2}(A - A')$, then

$$K' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}\left[A' - (A')'\right] = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -K$$

Hence, $K = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

3. Let $K = AA'$, then as $(AB)' = B'A'$, so

$$K = (AA')' = (A')'A' = AA' = K .$$

Hence K is symmetric.

Similarly $A'A$ is symmetric.

3.6.3. Theorem. A matrix which is both symmetric and skew symmetric, must be a null matrix.

Proof. Let A be a matrix which is both symmetric and skew symmetric. Then

$$A' = A \quad \text{and} \quad A' = -A$$

Subtracting these two, we obtain

$$2A = \mathbf{O}$$

which implies that A is a zero matrix

3.6.4. Theorem. Let *A* and *B* be symmetric matrices of same order, then

1. $A+B$ is a symmetric matrix.

2. $AB + BA$ is a symmetric matrix.

3. $AB - BA$ is a skew-symmetric matrix.

Proof.

Since A and B are symmetric matrices. So $A' = A$ and $B' = B$. Then, we have

1. $(A + B)' = A' + B' = A + B$

Hence A + B is symmetric.

2. $(AB + BA)' = (AB)' + (BA)' = B'A' + A'B' = BA + AB = AB + BA.$

Hence $AB + BA$ is symmetric matrix.

3. $(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$

Hence $AB - BA$ is skew symmetric matrix.

3.6.5. Theorem. Every square matrix can be uniquely expressed as the sum of a symmetric and skew symmetric matrix.

Proof. Let A be a square matrix , then we can write

$$A = \frac{1}{2}(A+A) = \frac{1}{2}(A + A' + A - A') = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q, \text{ say}$$

where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$. As proved earlier P is a symmetric matrix and Q is a skew symmetric matrix.

To prove the uniqueness. If possible, assume that

$$A = B + C \qquad (1)$$

where R is symmetric and S is skew symmetric.

Then, $$A' = (B + C)' = B' + C' = B - C \qquad (2)$$

Adding (1) and (2),

$$A + A' = 2B \qquad \Rightarrow \qquad B = \frac{1}{2}(A + A') = P.$$

Subtracting (2) from (1) ,

$$A + A' = 2C \qquad \Rightarrow \qquad C = \frac{1}{2}(A - A') = Q.$$

Hence A is uniquely expressed as a sum of symmetric and skew symmetric matrix.

3.6.6. Example. Find the transpose of matrix $A = \begin{bmatrix} 3 & 4 & 7 \\ 1 & 2 & 5 \\ -3 & 4 & 5 \end{bmatrix}$.

Solution. By definition, transpose of $A = A' = \begin{bmatrix} 3 & 1 & -3 \\ 4 & 2 & 4 \\ 7 & 5 & 5 \end{bmatrix}$.

3.6.7. Example. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, then verify that $(AB)' = B'A'$.

Solution. Given that $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Therefore, $AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 7 \\ 6 & 1 & -1 \end{bmatrix}$. Thus,

$$(AB)' = \begin{bmatrix} 8 & 6 \\ 3 & 1 \\ 7 & -1 \end{bmatrix}$$

Also, $B'A' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix}' \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 3 & 1 \\ 7 & -1 \end{bmatrix}$.

So, we observed that

$$(AB)' = B'A'$$

3.6.8. Example. Express matrix A= $A = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix}$ as a sum of symmetric and skew symmetric matrices.

Solution. We know that $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew symmetric.

Now , $A + A' = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 10 & 18 & 3 \\ 7 & 4 & 1 \\ 9 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 20 & 25 & 12 \\ 25 & 8 & -9 \\ 12 & -9 & 14 \end{bmatrix}$

and $\quad A - A' = \begin{bmatrix} 10 & 7 & 9 \\ 18 & 4 & -10 \\ 3 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 10 & 18 & 3 \\ 7 & 4 & 1 \\ 9 & -10 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -11 & 6 \\ 11 & 0 & -11 \\ -6 & 11 & 0 \end{bmatrix}$.

Now $\quad \frac{1}{2}(A + A') = \begin{bmatrix} 10 & \frac{25}{2} & 6 \\ \frac{25}{2} & 4 & -\frac{9}{2} \\ 6 & -\frac{9}{2} & 7 \end{bmatrix}$ which is symmetric.

and $\quad \frac{1}{2}(A - A') = \begin{bmatrix} 0 & -\frac{11}{2} & 3 \\ \frac{11}{2} & 0 & -\frac{11}{2} \\ -3 & \frac{11}{2} & 0 \end{bmatrix}$ which is a skew symmetric matrix.

Thus $A = A = \begin{bmatrix} 10 & \frac{25}{2} & 6 \\ \frac{25}{2} & 4 & -\frac{9}{2} \\ 6 & -\frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{11}{2} & 3 \\ \frac{11}{2} & 0 & -\frac{11}{2} \\ -3 & \frac{11}{2} & 0 \end{bmatrix}$.

3.6.9. Exercise.

1. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -5 & -3 \\ 2 & 5 & 3 \end{bmatrix}$ verify that $(A+B)' = A' + B'$.
2. Find the transpose of following matrices

   (i) $A = \begin{bmatrix} 5 & 2 & 0 \\ 1 & 4 & 7 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 11 & 9 \\ 2 & 7 & 4 \\ 5 & 6 & 7 \end{bmatrix}$
3. Find values of x, y, z for the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ if $A'A = I$.
4. Express the following matrices as the sum of symmetric and skew symmetric matrix.

   (i) $\begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 3 \\ 0 & 6 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 7 \\ 6 & 5 & -8 \\ 1 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 4 & 2 & -1 \\ 3 & 5 & 7 \\ 1 & -2 & 1 \end{bmatrix}$

3.7. Check Your Progress.

4. Give an example of a matrix which is row matrix as well as column matrix.
5. Find a matrix X such that $2A + B + X = 0$ where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.
6. If $A = \begin{bmatrix} 2 & -8 \\ 5 & -3 \end{bmatrix}$, then show that $A + A'$ is symmetric and $A - A'$ is skew symmetric.

Answers.

1. Any square matrix of order 1 is a matrix which is both row matrix as well as column matrix.
2. $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$.

3.8. Summary. In this chapter, we discussed about Matrices, its various types, when we can add or subtract or multiply two matrices. In all cases the most important aspect is the order of the given matrices. Further, it was observed that ant square matrix can be expressed as sum of a symmetric and a skew-symmetric matrix.

References.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.

4

Determinants

Structure

4.1. Introduction. In this chapter, we shall learn to evaluate the determinant of a square matrix and then with the help of this we will solve some system of linear equations having two or three variables.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Determinants.
(ii) Inverse of a matrix.
(iii) Applying row and column operations wherever required.
(iv) Solving system linear equations.

4.1.2. Keywords. Matrix, Determinant, Inverse of a Matrix, Adjoint of a matrix.

4.2. Determinants.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be a square matrix of order n. Then a unique number can be associated to A, known as its determinant. The determinant of A can be denoted by:

$$\det A \quad \text{or} \quad |A| \quad \text{or} \quad |a_{ij}| \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

1. If $A = (a_{11})_{1\times 1}$, then the determinant of A is defined as $|A| = a_{11}$.
2. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$, then determinant of A is defined as $|A| = \begin{vmatrix} a_{11} & & a_{12} \\ & \searrow \swarrow & \\ a_{21} & & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.
3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

4. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, then determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

For matrices of higher order similar procedure can be adopted.

4.2.1. Singular and Non-singular Matrices:

Any square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

4.2.2. Minors and Cofactors.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be any matrix, then minor of an element a_{ij}, denoted by M_{ij} is the determinant of elements of A obtained by removing ith row and jth column of A, keeping the order of rest rows and columns unchanged.

Thus, $M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$.

The cofactor of a_{ij}, denoted by A_{ij}, is defined to be $(-1)^{i+j}\ M_{ij}$.

For example, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2. Then, minors are obtained as

$$M_{11} = \text{Minor of } a_{11} = a_{22}, \qquad M_{12} = \text{Minor of } a_{12} = a_{21}$$

$$M_{21} = \text{Minor of } a_{21} = a_{12}, \qquad M_{22} = \text{Minor of } a_{22} = a_{11}$$

and cofactors are obtained by

$$A_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} . M_{11} = a_{22},$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^{1+2} . M_{12} = -a_{21},$$

$$A_{21} = \text{Cofactor of } a_{21} = (-1)^{2+1} . M_{21} = -a_{12},$$

$$A_{22} = \text{Cofactor of } a_{22} = (-1)^{2+2} . M_{22} = a_{11}.$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3. Then,

$$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\, a_{33} - a_{23}\, a_{31}$$

$$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\, a_{32} - a_{22}\, a_{31}.$$

Minors for remaining elements can be obtained in the similar pattern. Further,

$$A_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} M_{11} = M_{11} = (a_{22}\, a_{33} - a_{23}\, a_{32})$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{23}\, a_{31})$$

$$A_{13} = \text{Cofactor of } a_{13} = (-1)^{1+2} \ M_{13} = M_{13} = a_{21}\, a_{32} - a_{22}\, a_{31}.$$

Cofactors for remaining elements can be obtained in the similar pattern.

Remark. It should be noted that if A is any matrix, then its determinant is the sum of products of elements of any row and their corresponding cofactors. Thus,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in}.$$

4.2.3. Example. Solve for x:

$$\begin{vmatrix} 5x & 3 \\ 2 & 3 \end{vmatrix} = 5.$$

Solution. Here, $\begin{vmatrix} 5x & 3 \\ 2 & 3 \end{vmatrix} = 5 \quad \Rightarrow 15x - 6 = 5 \Rightarrow 15x = 11 \Rightarrow x = \dfrac{11}{15}.$

4.2.4. Example. Write the minors and co-factors of all the elements in $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 2 \\ 5 & 1 & 3 \end{pmatrix}$.

Solution. Let M_{ij} and A_{ij} denotes the minor and co-factor of the element a_{ij} respectively, then

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 0 - 2 = -2 \text{ and } A_{11} = (-1)^{1+1} M_{11} = -2.$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} = 9 - 10 = -1 \text{ and } A_{12} = (-1)^{1+2} M_{12} = 1.$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3 - 0 = 3 \text{ and } A_{13} = (-1)^{1+3} M_{13} = 3.$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 0 - 2 = -2 \text{ and } A_{21} = (-1)^{2+1} M_{21} = 2.$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 3 - 10 = -7 \text{ and } A_{22} = (-1)^{2+2} M_{22} = -7.$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} = 1 - 0 = 1 \text{ and } A_{23} = (-1)^{2+3} M_{23} = -1.$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0 - 0 = 0 \text{ and } A_{31} = (-1)^{3+1} M_{31} = 0.$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4 \text{ and } A_{32} = (-1)^{3+2} M_{32} = 4.$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0 - 0 = 0 \text{ and } A_{33} = (-1)^{3+3} M_{33} = 0.$$

4.2.5. Example. If $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{bmatrix}$ find $|A|$ by expanding along first row and second column and verify that the value is same.

Solution. Expanding by first row, we have

$$|A| = 1(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1(-3) \; -2(-6) \; +3(-3) = 0.$$

Again, expanding by second column, we have

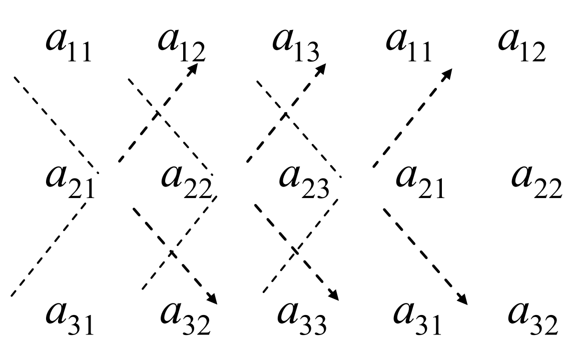
$$|A| = 2(-1)^{2+1}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -2(36-42)+5(9-21)-8(6-12) = 0.$$

Thus the determinant obtained by expanding along different rows are same.

4.2.6. Determinant using Sarrus Method.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

First write five columns in the following order:



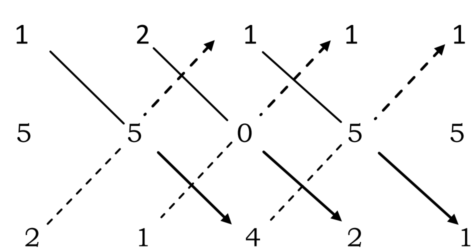
The value of $|A|$ is given by adding the products of the diagonals going from top to bottom and subtracting the products of the diagonals going from bottom to top. Thus

$$|A| = \left(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right) - \left(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}\right)$$

Note. Sarrus Method is used only for determinant of order 2 and 3.

4.2.7. Example. Evaluate the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 5 & 5 & 0 \\ 2 & 1 & 4 \end{vmatrix}$ using Sarrus Method.

Solution. By Sarrus diagram,



we have, $|A| = (1.5.4 + 2.0.2 + 1.5.1) - (2.5.1 + 1.0.1 + 4.5.1)$

$= 25 - 30 = -5.$

4.2.8. Exercise.

1. Which of the following matrices are singular and which are non-singular.

(i) $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 \\ 0 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 4 & 3 & 7 \end{bmatrix}$

2. For what value of λ, the matrix $\begin{bmatrix} 7 & 1 \\ 2 & \lambda \end{bmatrix}$ is singular.

3. Find the minors and cofactors the following matrices:

(i) $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 1 \\ 2 & 3 \end{bmatrix}$

4. Solve the following equations for x:

(i) $\begin{vmatrix} 3x & 4 \\ 0 & 2 \end{vmatrix} = 8$ (ii) $\begin{vmatrix} x & x \\ -5 & x \end{vmatrix} = -6$

5. Find the following determinants.

(i) $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

6. Find the determinant using Sarrus Method: $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix}$

Answer.

1. (i) Singular. (ii) *N*on-singular. (iii) *N*on-singular.

2. $\lambda = \dfrac{2}{7}$.

3. (i) $M_{11} = -1,\ M_{12} = 2,\ M_{21} = -1\ ,\ M_{22} = 1\ ,\ A_{11} = -1,\ A_{12} = -2,\ A_{21} = 1,\ A_{22} = 1$

(ii) $M_{11} = 3,\ M_{12} = 2,\ M_{21} = 1\ ,\ M_{22} = 7\ ,\ A_{11} = 3,\ A_{12} = -2,\ A_{21} = -1,\ A_{22} = 7$

4. (i) $\dfrac{4}{3}$ (ii) $x = -3\ , -2$

5. (i) – 7 (ii) – 9 (iii) 4*abc*

6. – 9

4.3. Properties of Determinants.

Using the following properties of determinants, we can evaluate the determinant of a matrix without using the evaluation methods discussed earlier.

We will use the notations $R_1, R_2, ..., C_1, C_2, ...$ to denote row one, row two, …, column one, column two, ... etc. of a matrix.

1. The value of determinant remains unchanged if rows (columns) are changed into columns (rows), that is, if A is a matrix, then $|A| = |A'|$.
2. If two adjacent rows (columns) of a determinant are interchanged then the value of determinant is multiplied by -1.
3. If any two rows (columns) are identical then the value of the determinant is zero.
4. If any two rows (columns) are multiples of each other then the determinant is zero.
5. If all entries of any row (column) are zero then the determinant is zero.
6. If each element in a row (column) of a determinant is multiplied by any scalar then the determinant is also multiplied by same scalar.
7. If every element of any row (column) is the sum (or difference) of two or more quantities, then the determinant can also be expressed as the sum (difference) of two or more determinants of same order.

   For example, let $\Delta = \begin{vmatrix} 7 & 2 & 1 \\ 4 & 5 & 2 \\ 3 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5+2 & 2 & 1 \\ 3+1 & 5 & 2 \\ 2+1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 5 & 2 \\ 2 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 1 \\ 1 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix}$

8. If to every element of a row (column) of a determinant be added or subtracted equal multiples of the corresponding elements of one or more rows (or columns) then the value of the determinant unchanged.
9. The determinant of product of square matrices of same order is equal to the product of the determinants of matrices, that is, $|AB| = |A|.|B|$

4.3.1. Example. Without expanding show that following determinant vanishes.

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 1 & 1 & 8 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 29 & 1 & 4 \\ 33 & 5 & 4 \\ 17 & 3 & 2 \end{vmatrix}$

Solution. (i) Let $\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 1 & 1 & 8 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and using property 5,we get

$$\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2-2 & 6-6 & 10-10 \\ 1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 1 & 1 & 8 \end{vmatrix} = 0$$

(ii) Let
$$\Delta = \begin{vmatrix} 29 & 1 & 4 \\ 33 & 5 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 7C_3$ and using property 3, we get

$$\Delta = \begin{vmatrix} 29-28 & 1 & 4 \\ 33-28 & 5 & 4 \\ 17-14 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 5 & 5 & 4 \\ 3 & 3 & 2 \end{vmatrix} = 0$$

4.3.2. Example. Using properties of determinants, show that $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2 - bc} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2 - ca} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0}$.

Solution : Let $\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ then

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Multiplying R_1,R_2 and R_3 of second term of Δ by a, b and c, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_2$ in second term of Δ, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

4.3.3. Example. Show that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution : Let

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + 0$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 2a & 6a & 3c \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2 c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \,.\, 0 + a^2 c \,.\, 0$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} = a^3 \times 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = a^3(7-6) = a^3$$

4.3.4. Example. Evaluate $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$.

Solution: Let $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

as first and third rows are identical.

4.3.5. Example. Show that $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

Solution. Let $\Delta = \begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by a, b, and c respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} (b+c)^2 a & ba^2 & ca^2 \\ ab^2 & (c+a)^2 b & cb^2 \\ ac^2 & bc^2 & (a+b)^2 c \end{vmatrix}$$

Taking a, b and c common from C_1,C_2 and C_3, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

Taking $a+b+c$ common from C_1 and C_2, we get

$$\Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Applying $C_1 \to C_1(a)$, $C_2 \to C_2(b)$

$$\Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3$

$$\Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Taking a, b and $2ab$ common from R_1, R_2 and R_3 respectively

$$\Delta = \frac{(a+b+c)^2}{ab}\ ab\ .\ 2ab\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

Now expanding along R_3, we get

$$\Delta = 2ab(a+b+c)^2\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} = 2ab(a+b+c)^2\ [(b+c)(c+a)-ab) = 2abc(a+b+c)^3$$

4.3.6. Example. Show that $\begin{vmatrix} \boldsymbol{b+c} & \boldsymbol{c+a} & \boldsymbol{a+b} \\ \boldsymbol{q+r} & \boldsymbol{r+p} & \boldsymbol{p+q} \\ \boldsymbol{y+z} & \boldsymbol{z+x} & \boldsymbol{x+y} \end{vmatrix} = \mathbf{2}\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{p} & \boldsymbol{q} & \boldsymbol{r} \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{vmatrix}$.

Solution : Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

4.3.7. Exercise.

1. Without expanding show that following determinant vanishes.

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$ (ii) $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$

(v) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & a & abc \\ 1 & b & abc \\ 1 & c & abc \end{vmatrix}$ (vii) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & \mathrm{a} & \mathrm{abc} \\ 1 & \mathrm{b} & \mathrm{abc} \\ 1 & \mathrm{c} & \mathrm{abc} \end{vmatrix}$

2. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$

3. Show that

(i) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$ (ii) $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(iii) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$ (iv) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$

(v) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

4. Show that

(i) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$ (ii) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

5. Show that

(i) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$ (ii) $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$

4.4. Adjoint of a Matrix.

Let $A = \left[a_{ij}\right]_{n\times n}$ be a square matrix. Then the adjoint of matrix A is defined as

$$\mathrm{adj}A = [A_{ij}]'$$

where A_{ij} is the corresponding co-factor of a_{ij}.

4.4.1. Example. Find the adjoint of matrix $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

Solution. Given that $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$.

By definitions of Cofactors:

$$A_{11} = \text{cofactor of } a_{11} = 4$$

$$A_{12} = \text{cofactor of } a_{12} = -3$$

$$A_{21} = \text{cofactor of } a_{21} = -2$$

$$A_{22} = \text{cofactor of } a_{22} = 1$$

Thus, $\quad \text{adj } A = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} = \begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}' = \begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$.

4.4.2. Theorem. If A is square matrix of order $n\times n$, then prove that

$$A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)A.$$

4.4.3. Example. Find adjoint of $A=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$ and also verify that $(\text{adj}A)A = |A|\,I_2 = A(\text{adj}\,A)$.

Solution : Given that $A=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$

Cofactors of elements of A are:

$A_{11} = \text{cofactor of } a_{11} = 5$, $\qquad A_{12} = \text{cofactor of } a_{12} = -3$

$A_{21} = \text{cofactor of } a_{21} = -2$, $\qquad A_{22} = \text{cofactor of } a_{22} = 1$

Thus, $\text{adj}A = \begin{bmatrix}5 & -3\\ -2 & 1\end{bmatrix}' = \begin{bmatrix}5 & -2\\ -3 & 1\end{bmatrix}$

Now $\qquad |A| = \begin{vmatrix}1 & 2\\ 3 & 5\end{vmatrix} = 5-6=-1$

So $\qquad A(\text{adj}A) = \begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}\begin{bmatrix}5 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = |A|\,I_2$

Again $\;(\text{adj}AA) = \begin{bmatrix}5 & -2\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = |A|\,I_2$

So, we get

$$A(\text{adj } A) = |A|\,I_2 = (\text{adj } A)\,A$$

4.4.4. Exercise.

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ verify, adj(AB) = (adj$B$) (adj$A$).

2. Find the adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$. Also show that $A.(\text{adj } A) = |A|.I_3 = (\text{adj } A).A$.

3. Find the adjoint of following matrices.

   (i) $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4.5. Inverse of a Matrix.

A square matrix of order n is invertible if there exist a square matrix B of same order such that $AB = I_n = BA$.

In such a case, we say that inverse of A is B and inverse of B is A and we write

$$A^{-1} = B, B^{-1} = A .$$

If inverse of a matrix exists, then it is called an invertible matrix.

4.5.1. Theorem. A square matrix is invertible iff it is non-singular.

Proof. Let A be an invertible matrix. Then, there exists a matrix B such that

$$AB = I_n = BA$$

$\Rightarrow$ $|AB| = |I_n|$

$\Rightarrow$ $|A||B| = 1$

$\Rightarrow$ $|A| \neq 0$

$\Rightarrow$ A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n that is, $|A| \neq 0$. Then, we know that

$$A(\text{adj } A) = |A| I_n = (\text{adj } A)A$$

Dividing both sides by $|A|$,

$$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right)A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A$$

Hence, A is an invertible matrix.

Remark. Due to the above theorem, we can say that the inverse of a non-singular matrix A is given by

$$A^{-1} = \frac{adj\ A}{|A|}$$

4.5.2. Theorem. If *A* is an invertible square matrix, then A' is also invertible and

$$(A')^{-1} = \left(A^{-1}\right)'.$$

Proof. Since *A* is an invertible matrix, so $|A| \neq 0$, and thus $|A'| \neq 0$, which implies A' is also invertible.

Now, $$AA^{-1} = I_n = A^{-1}A$$

$$\Rightarrow \quad \left(AA^{-1}\right)' = (I_n) = \left(A^{-1}A\right)'$$

$$\Rightarrow \quad \left(A^{-1}\right)'(A') = I_n = A'\left(A^{-1}\right)'$$

$$\Rightarrow \quad (A')^{-1} = \left(A^{-1}\right)'$$

4.5.3. Theorem. If *A* and *B* are invertible matrices of the same order, then so is *AB* and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. It is given that *A* and *B* are invertible matrices, therefore $|A| \neq 0$ and $|B| \neq 0$

$$\Rightarrow \quad |A||B| \neq 0$$

$$\Rightarrow \quad |AB| \neq 0$$

$\Rightarrow$ *AB* is a invertible matrix.

Now,

$$(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n$$

and,

$$\left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}(I_nB) = B^{-1}B = I_n$$

Thus, $$(AB)\left(B^{-1}A^{-1}\right) = I_n = \left(B^{-1}A^{-1}\right)(AB)$$

Hence, $$(AB)^{-1} = B^{-1}A^{-1}.$$

4.5.4. Theorem. Inverse of an invertible matrix is always unique.

Proof. Let *A* be an invertible matrix of order $n \times n$ having matrices *B* and *C* as its two inverses. Then,

$$AB = BA = I_n \text{ and } AC = CA = I_n$$

Now, $AB = I_n$

$$\Rightarrow \quad C(AB) = CI_n$$

$$\Rightarrow \quad (CA)B = CI_n$$

$$\Rightarrow \quad I_nB = CI_n$$

$$\Rightarrow \quad B = C$$

Hence, inverse of a matrix is unique.

4.5.5. Corollary. If A is an invertible matrix, then $(A^{-1})^{-1} = A$.

Proof. We have,

$$AA^{-1} = I = A^{-1}A$$

$\Rightarrow$ A is the inverse of A^{-1}, that is, $A = (A^{-1})^{-1}$.

4.5.6. Example. Find the inverse of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Solution : Given that $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Therefore, $|A| = 1+1 = 2 \neq 0$, which implies A^{-1} exists.

Now, by definition

$$A_{11} = \text{cofactor of } a_{11} = 1$$
$$A_{12} = \text{cofactor of } a_{12} = 1$$
$$A_{21} = \text{cofactor of } a_{21} = -1$$
$$A_{22} = \text{cofactor of } a_{22} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now $A^{-1} = \dfrac{1}{|A|} \text{ adj } A = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

4.5.7. Example. If $A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$, show that $A^2 - 6A + I = O$. Hence find A^{-1}.

Solution. Here, $A^2 = A.A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix}$

So $A^2 - 6A + I = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix} - 6\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix} - \begin{bmatrix} 12 & 42 \\ 6 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence, $A^2 - 6A + I = O$.

Now using this we have to find A^{-1}.

$$A^2 - 6A + I = O \Rightarrow \qquad 6A - A^2 = I$$

Now pre-multiplying both sides by A^{-1} we have,

$$A^{-1} = 6I - A$$

So, $A^{-1} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$.

4.5.8. Exercise.

1. Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify your answer.

2. For the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{6} & \mathbf{7} \\ \mathbf{8} & \mathbf{9} \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that

$$aA^{-1}=(a^2+bc+1)I_2-aA.$$

4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2-4A-5I = O$ and hence find A^{-1}.

4.6. Inverse of a Matrix by using Elementary Operations.

4.6.1. Elementary Operations. To obtain inverse of a matrix sometimes we use some operations on a given matrix called elementary operations.

These are of two types:

1. **Elementary row operations.** Elementary operation on rows of a matrix are known as elementary row operation. Following are the various types of elementary row operations

i) The interchange of any two rows. By $R_i \leftrightarrow R_j$, we mean interchanging ith row of the given matrix with jth row.

ii) The multiplication of the elements of row by a non-zero number. By $R_i \to k R_i$, we mean that the elements of ith row of the given matrix are multiplied by k.

iii) Adding to the elements of a row, the corresponding elements of any other row multiplied by any scalar *k*. By $R_i \to R_i + kR_j$, we mean that the elements of jth row of the given matrix are multiplied by k and then the elements are added to corresponding elements of ith row.

Remark. An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor.

4.6.2. To find inverse of a square matrix by using elementary row operation.

Let A be a non-singular matrix. So, it can be written as $A = IA$, where I is identity matrix. Now apply elementary row operations on A to convert it to I and on right side apply these operations as applied on left side to I. If I is converted to B, then this matrix B is inverse of A.

2. **Elementary column Operations.** The similar operations are defined for columns and known as elementary column operations. Also to find inverse of a matrix A this time we will consider $A = AI$ and then apply elementary columns operations on A to convert it to I and on right side apply these operations as applied on left side to I. If I is converted to B, then this matrix B is inverse of A.

4.6.3. Example. Find inverse using elementary row operations of $A=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$.

Solution : Given that $A=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$, then $|A|=\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}=4-3\neq 0$. So A^{-1} exists.

Now let $A = IA$, which implies $\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}A$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 - 3R_2$, we get

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}A$$

Therefore, $A^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$.

4.6.4. Example. Find the inverse of matrix $\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$ using elementary column operation.

Solution. Clearly A is invertible.

Now let $A = AI$, which implies $\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}=A\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

Applying $C_2 \to C_2 - 3C_1$, we get

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=A\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=A\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$$

Therefore, $A^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$.

4.6.5. Exercise.

1. Find the inverse of matrix $A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$ by using elementary row operations.
2. Find the inverse of the matrix $A=\begin{bmatrix}1 & 0 & 0\\ 3 & 2 & 0\\ 1 & 1 & 2\end{bmatrix}$ using elementary row operations.

3. Using elementary column operations, find the inverse of matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$.

4. Find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ by using elementary column operations.

4.7. Solution of Simultaneous Linear Equations.

A system of linear equation has either unique solution or infinitely many solutions or no solution. If a system of linear equations has a solution (whether unique or infinite), then the system is said to be **consistent** and if the system has no solution, it is said to be **inconsistent**.

4.7.1. Cramer's Rule to Solve the Linear Equations.

1. System of Linear Equation of two variables *x* and *y*.

First we consider a system of linear equations in two variables x and y:

$$ax + by = d_1$$
$$cx + dy = d_2$$

We define D as the determinant obtained from the coefficients of x and y, D_1 and D_2 are determinants obtained by replacing first and second column respectively of D by $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Thus,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix},\ D_1 = \begin{vmatrix} d_1 & b \\ d_2 & d \end{vmatrix},\ D_2 = \begin{vmatrix} a & d_1 \\ c & d_2 \end{vmatrix}$$

If $D \neq 0$, then the system has a unique solution given by

$$x = \frac{D_1}{D},\quad y = \frac{D_2}{D}.$$

2. System of Linear Equation of two variables *x, y* and *z*.

Now we consider a system of linear equations in three variables x and y and z:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then as defined in case of two variables, we define the following:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $D \neq 0$, then the system has unique solution and given by

$$x = \frac{D_1}{D},\quad y = \frac{D_2}{D},\quad z = \frac{D_3}{D}$$

Remark. If $D = 0$, then the system has either infinitely many solutions or no solution. However, the systems with such solutions are not included in the syllabi.

4.7.2. Example. Solve the following system of equations using Cramer's Rule

$$x + y = 5$$
$$x + 2y = 15$$

Solution. Given system of equations is

$$x + y = 5$$
$$x + 2y = 15$$

Then, by definition

$$D = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

Therefore, the system has a unique solution.

Now $$D_1 = \begin{vmatrix} 5 & 1 \\ 15 & 2 \end{vmatrix} = 10 - 15 = -5$$

and $$D_2 = \begin{vmatrix} 1 & 5 \\ 1 & 15 \end{vmatrix} = 15 - 5 = 10.$$

Then, by Cramer's Rule, the unique solution is given by

$$x = \frac{D_1}{D} = -\frac{5}{1} = -5, \qquad y = \frac{D_2}{D} = \frac{10}{1} = 10.$$

So, $x = -35$, $y = 25$ is a solution.

4.7.3. Exercise. Solve the following system of equations by using Cramer's Rule:

1. $$\begin{aligned} x + y + z &= 1 \\ 3x + 5y + 6z &= 4 \\ 9x + 2y - 36z &= 17 \end{aligned}$$

2. $$\begin{aligned} 2y - 3z &= 0 \\ x + 3y &= -4 \\ 3x + 4y &= 3 \end{aligned}$$

3. $$\begin{aligned} 2x + 3y &= 7 \\ 4x - 5y &= 3 \end{aligned}$$

4. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to it, we get 7. By adding second and third numbers to three times the first number, we get 12. Find the numbers.

5. The perimeter of a triangle is 45 cm. The longest side exceeds the shortest side by 8 cm and sum of the length of the longest and the shortest side is twice the length of the other side. Find the lengths of sides of the triangle.

6. Find *a*, *b*, *c* when $f(x) = ax^2 + bx + c$, $f(1) = 1$, $f(2) = 2$, $f(0) = 4$. Determine the quadratic function $f(x)$ and find its value when $x = 0$.

Answers.

1. $x = \frac{1}{3}$, $y = 1$, $z = -\frac{1}{3}$
2. $x = 5, y = -3, z = -2$
3. $x = 2$, $y = 1$
4. 3,1,2
5. 19 cm, 15 cm, 11 cm
6. $2x^2 - 5x + 4$, 4

4.7.4. Matrix Method to solve system of linear equations.

1. System of Linear Equation of two variables *x* and *y*.

First we consider a system of linear equations in two variables x and y:

$$a_1x + b_1y = d_1$$

$$a_2x + b_2y = d_2$$

We define $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Then the given system of equations can be written in matrix form as

$$AX=B.$$

If $|A| \neq 0$, then the system has unique solution given by

$$X = A^{-1}B.$$

2. System of Linear Equation of three variables *x, y* and *z*.

First we consider a system of linear equations in two variables x, y and z:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Define $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

If $|A| \neq 0$, then the system has unique solution given by

$$X = A^{-1}B$$

Remark. If $|A| = 0$, then the system has either infinitely many solutions or no solution. However, the systems with such solutions are not included in the syllabi.

4.7.5. Example. Solve the following system of equations by matrix method:

$$x + y = 1$$
$$2x + y = 2$$

Solution. The given system of equations can be represented in matrix form as $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now, $|A| = 1 - 2 = -1 \neq 0$.

Thus, the system has a unique solution given by

$$X = A^{-1}B$$

We need to obtain the inverse of A, for this cofactors of elements of A are

$$A_{11} = 1 \quad , \quad A_{12} = -2 \quad , \quad A_{21} = -1 \quad , \quad A_{22} = 1.$$

Thus, $$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

and $$A^{-1} = \frac{1}{|A|}\text{adj } A = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Therefore, the solution can be obtained from

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence $x = 2$, $y = -1$ is a solution.

4.7.6. Exercise. Solve the following system of equations:

1. $$\begin{aligned} 2x + 8y + 5z &= 6 \\ x + y + z &= -2 \\ x + 2y - z &= 2 \end{aligned}$$

2. $$\begin{aligned} \frac{2}{x} - \frac{3}{y} + \frac{3}{z} &= 10 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 10 \\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} &= 13 \end{aligned}$$

Answers.

1. $x = -3, \quad y = 2, \quad z = -1$
2. Use $\frac{1}{x} = u$, $\frac{1}{y} = v$, $\frac{1}{z} = w$, then solving the system we will obtain $u = 2$, $v = 3$, $w = 5$.

4.8. Check Your Progress.

1. Write the minors and cofactors of all elements of $\begin{bmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{bmatrix}$

2. For the matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$. Hence find A^{-1}.

Answers.

1. $M_{11} = -2, M_{12} = -7, M_{13} = 3, M_{21} = 5, M_{22} = 7, M_{23} = -11, M_{31} = 4, M_{32} = 7, M_{33} = -6$

$A_{11} = -2, A_{12} = 7, A_{13} = 3, A_{21} = -5, A_{22} = 7, A_{23} = 11, A_{31} = 4, A_{32} = -7, A_{33} = -6$

4.9. Summary. In this chapter, we discussed about determinants of matrices, invertible matrices and the role played by an invertible matrix to solve a system of linear equations having a unique solution.

References.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.